A Model for Bio-Economics of Fisheries

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Abstract. In this paper a model for growth of fish, a model for fishing economics and delay model for fishing are considered. The maximum sustainable yield for fishing is obtained. In the delay model the three cases of equilibrium population being equal to (or) greater then (or) less then the ratio of carrying capacity and rate of growth are considered.

1 Introduction

The World population is growing at enormous rate, creating increasing demand for food. Food comes from renewable resources. Agricultural products are renewable resources, since every season new crops are produced in farms. Fisheries are a renewable resource since fish are reproduced in lakes and seas. Forests are renewable resources since they reproduce periodically. As these resources are renewable, the quality of the resources will certainly degrade, leading to shortage. Over fishing will lead to decline in fisheries. Global warming again has an impact on the growth of agriculture, fisheries and forests. It is imperative that we should manage these resources economically to prevent a catastrophic bust in our global economy.

Mathematical bio-economics is the mathematical study of the management of renewable bio resources. It takes into consideration not only economic factors like revenue, cost etc., but also the impact of this demand on the resources.

One of the mathematical tools used in bio economics is differential equations. To model the growth or decline of fish population, we use differential equations.

2 Model for Growth of Fish

As a first step in the model of growth of fisheries, we assume that ***"Fish reproduce at a rate that is linearly dependent on the number of fish present at time t".***

We take $r = b - d > 0$ where b is the birth rate and d is the death rate, both being constant. Then the model for the growth of fish population is

$$\frac{dN}{dt} = rN \qquad (1)$$

where $N(t)$ is the population size of the fish at time t and r is the reproduction rate of fish, taken to be constant. Let the initial population of the fish be N_0, that is

$$N(0) = N_0 \qquad (2)$$

Solving (1) with the initial condition (2) we have

$$N(t) = N_0 e^{rt} \qquad (3)$$

The model (1) and (2) or its solution (3) is called Malthusian growth model. We can interpret this as follows. Initially there is a population of size $N_0 (> 0)$, however small it may be; then the population grows exponentially to the size $N(t) = N_0 e^{rt}$ in time t. (Malthus predicted a doom's day for the human population using the above model). This model is not realistic as for as the growth of fisheries is concerned. To add realism to this model we have to consider the effect of crowding of fish, limitation of space and resources. We may call it *carrying capacity*. Such an effect is called negative density dependence denoted by $-\frac{rN^2}{K}$ where K is the carrying capacity of the region (Lake).

Then the model for the growth of the fish can be written as

$$\frac{dN}{dt} = rN - \frac{rN^2}{K}$$
$$= rN\left(1 - \frac{N}{K}\right) \qquad (4)$$

The first term in the RHS of (4) is a linear (Malthusian) term and the second term $-\frac{rN^2}{K}$ is a nonlinear term which represents crowding effect. We can solve the equation (4) by separating the variable using partial fraction. But this solution does not give any; insight in the growth process of fish population. We have to undertake stability analysis of the model (4). Equate the RHS of (4) to zero, that is

$$rN\left(1 - \frac{N}{K}\right) = 0$$

Then $N = 0$ and $N = K$ are two equilibrium states.
Consider the zero equilibrium state $N = 0$
Put $N = 0 + \eta$ then (4) becomes

$$\frac{d\eta}{dt} = r\eta - \frac{r\eta^2}{K}$$
$$\frac{d\eta}{dt} \approx r\eta \qquad \text{(neglecting } \eta^2\text{)} \qquad (5)$$

This gives

$$\eta = ce^{rt} \to \infty \quad \text{as} \quad t \to \infty \qquad (6)$$

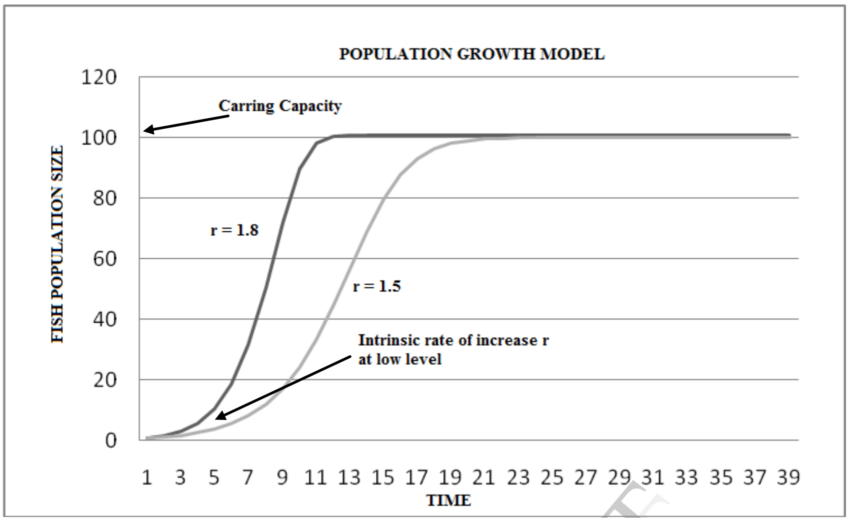
Therefore $N = 0$ is an unstable equilibrium state, which means that the fish population grows exponentially.
Now consider the non-zero equilibrium state $N = K$. Put $N = K + \eta$ where $|\eta| << 1$ then (4) becomes

$$\frac{d\eta}{dt} = r(K+\eta)\left[1-\frac{K+\eta}{K}\right] = -r\eta - \frac{r\eta^2}{K}$$

$$\frac{d\eta}{dt} \approx -r\eta \qquad (7)$$

$$\eta = ce^{-rt} \to 0 \quad \text{as} \quad t \to \infty .$$

Thus $N = K$ is a stable equilibrium state. Hence we conclude that after a long time the fish population will tend to the size K, the carrying capacity of the Lake.



3 A Model for Fishing Economics

In the previous section we discussed the model for the *bio growth of fish.* Now we consider the economics of fish growth. We know that the profit in any business is governed by the law:

Profit = Revenue – Costs

The total revenue is determined by

Total revenue = (Price of each resource harvested) × (Total number of resources (Yield))

Yield = qEN (8)

where q is called catchability coefficient which represents environmental factors, the ability to use location equipment to catch fish ($q > 1$), E is the effort exerted in harvesting the fish (depending on the number of boats used etc.,)

If the market price for the resources is p then the total revenue is given by

Total revenue = Price × Yield

$= pqEN$ (9)

The total cost of harvesting the resources is proportional to the effort exerted. That is

Total cost $= cE$,

where c is the constant which represents external control on cost (such as price of gasoline used in the power motor).

Thus the profit of harvesting the resources is given by

Economic rent = $pqEN - cE$

Harvesting the resource will reduce its abundance hence we have to subtract the yield qEN from the RHS of the equation (4) and we get

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - qEN$$

where $-qEN$ is the harvesting term.

Thus the bio economic model that governs the abundance of resource (fish) and the profit made from the resource is given by

$$\frac{dN}{dt}=rN\left(1-\frac{N}{K}\right)-qEN \qquad (10)$$

and

$$r-\frac{rN}{K}-qE=0$$

Economic rent = $pqEN - cE$ (11)

Equilibrium states of (10) are given by

$$rN\left(1-\frac{N}{K}\right)-qEN=0$$

$$N\left(r-\frac{rN}{K}-qE\right)=0$$

Thus we have to solve the equation.

Therefore $N=K-\frac{qEK}{r}$ (12)

The economic rent is the difference between total revenue and total cost. If costs exceed revenue, then the people will leave the resource because their efforts are not successful. Similarly, if the revenue is greater than the cost, the effort of expanded harvesting the resource will increase.

To calculate the yield of the resources put the expression $N=K-\frac{qEK}{r}$ in yield.

$$\text{Yield}\quad Y(E)=qEN$$

$$=qE\left(K-\frac{qEK}{r}\right)$$

$$=\frac{qK}{r}\left(E-qE^{2}\right) \qquad (13)$$

Note that this function is quadratic in E. The *maximum sustainable yield* (MSY) is found by maximizing Y(E). This occurs at the value of E where

$$E_{MSY}=\frac{r}{2q} \qquad (14)$$

We solve the system of equations when both economic rent and $\frac{dN}{dt}$ are equal to zero.

$$pqE^{*}N^{*}-CE^{*}=0$$

Then from (10) and (11)

$$rN^{*}\left(1-\frac{N^{*}}{K}\right)-qE^{*}N^{*}=0$$

We have

$$N^{*}=\frac{c}{pq},\qquad E^{*}=\frac{r}{q}\left(t-\frac{c}{pqK}\right) \qquad (15)$$

or $\quad N^{*}=0,\quad \mathrm{E}^{*}=0$

or $\quad N^{*}=K,\quad \mathrm{E}^{*}=0$

We can combine (15) and write

$$(x^*, E^*) = \begin{cases} \left(\frac{c}{pq}, \frac{r}{q}\left(1 - \frac{c}{pqK}\right)\right) \\ (0,0) \\ (K,0) \end{cases} \qquad (16)$$

Therefore maximum sustainable yield is

$$Y(E_{MSY}) = \frac{qk}{r}\left[\frac{r^2}{2q} - \frac{qr^2}{4qz}\right]$$

$$= \frac{kr}{4} \qquad (17)$$

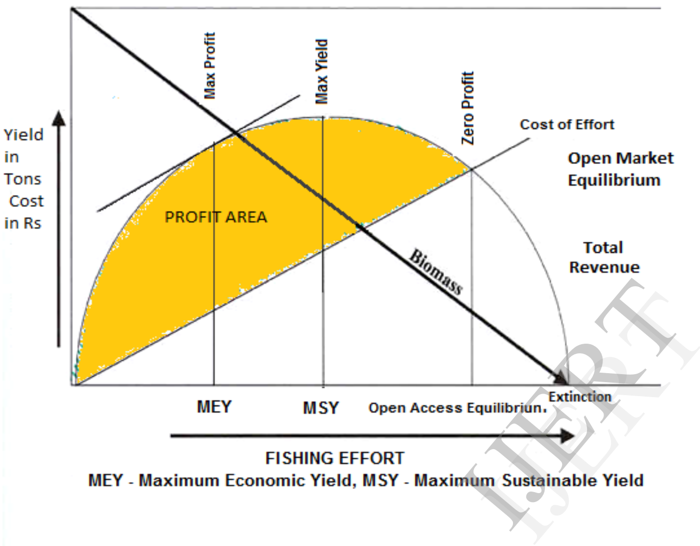


Figure – 2 Bio-economic Model of Fisheries in Equilibrium

[The arched curve represents total revenue from fishing effort in equilibrium. At a certain point, the number of fish being taken actually diminishes, even though fishing effort increases. It would technically be possible to take every fish out of the Lake. The straight line (Cost of Effort) represents total cost per fishing effort. The vertical distance between the total revenue and the total cost curve is economic profit. The maximum quantity of fish that could be removed annually is represented by the maximum sustainable yield.]

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4 A Delay Model

We consider the model equation for fishing when there is time delay of T to reach maturity, the finite gestation period.

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{N(t-T)}{K}\right] - qEN(t) \qquad 18)$$

where r is the rate of growth of the fish population, K is the carrying capacity of the region, q is the catchability coefficient, E is the effort exerted in harvesting the resource.

Substituting $N(t) = N^* + \eta(t)$ (19)

where N^* is an equilibrium state in (18), we get

$$\frac{d\eta}{dt} = r\left(N^* + \eta(t)\right)\left[1 - \frac{N^* + \eta(t-T)}{K}\right] \qquad - qE\left(N^* + \eta(t)\right)$$

$$\approx \frac{-rN^*\eta(t-T)}{K}$$

Thus the approximation equation is $\frac{d\eta}{dt} = -\frac{rN^*\eta(t-T)}{K}$ (20)

We take solution of (20) in the form $\eta(t) = ce^{\lambda t}$ (21)

from which we get $\lambda = \frac{-rN^* e^{-\lambda T}}{K} \qquad (T > 0)$ (22)

Analytic solution λ for (22) is difficult to be obtained. We will examine whether there are any solutions of (22) with real part Re λ>0 corresponding to which there is instability;

That is $\eta(t) \to \infty \quad as \quad t \to \infty$

For this put $\lambda = \mu + i\omega$

There exists a real number μ_0 such that all solutions λ of equation (22) satisfy Re $\lambda < \mu_0$

To examine this consider

$$|\lambda| = \frac{rN^*}{K}\left|e^{-(\mu+i\omega)T}\right|$$

$$= \frac{rN^*}{K}e^{-\mu t} \qquad \text{If} \quad |\lambda| \to \infty, \quad e^{-\mu t} \to \infty \qquad (23)$$

Which requires that $\mu \to -\infty$. Thus there must be a number μ_0 such that Re $\lambda < \mu_0$

Let us introduces $\frac{1}{\lambda} = z$ (24)

Then $w(z) = 1 + ze^{-\frac{T}{z}}$ (25)

Thus $w(z)$ has an essential singularity at z = 0, Then by Picard's theorem in complex analysis, in the neighborhood of z = 0, $w(z)$ has infinitely many roots λ of (22).

Now substituting $\lambda = \mu + i\omega$ in (22) we have $\mu + i\omega = \frac{-rN^*}{K}e^{-(\mu+i\omega)T}$

$$= \frac{-rN^*}{K}e^{-\mu T}\left[\cos\omega T - \sin\omega T\right] \qquad (26)$$

The real and imaginary parts of (26) are

$$\mu = -rN^* e^{-\mu T}\cos\omega T \qquad (27)$$

$$\omega = \frac{-rN^*}{K}e^{-\mu T}\sin\omega T \qquad (28)$$

We will now determine the range of T such that $\mu < 0$. That is, we will find conditions such that the upper limit $\bar{\mu}(of \ \mu)$ is negative.

We will first consider the simple case where λ is real. That is then equation (22) becomes

$$\lambda = \frac{-rN^*}{K} e^{-\mu T} \quad (29)$$

This has no positive root, since $e^{-\mu T} > 0$ for all μT

Next we consider $\omega \neq 0$. Then from equation (27) and (28), if ω is a solution of (27) and (28), $-\omega$ is also a solution.

Suppose $\omega > 0$ (with out loss of generality); then from equation (27), $\mu < 0$ requires $\omega T < \frac{\pi}{2}$ for all μT. Multiplying (28) by T

we get $\frac{TrN^*}{K} e^{-\mu T} \sin \omega T = \omega T < \frac{\pi}{2} \ldots (30)$

Case(i) If $N^* = \frac{K}{r}$ then $Te^{-\mu T} \sin \omega T < \frac{\pi}{2} \Rightarrow 0 < \mathrm{T} < \frac{\pi}{2}$ (31)

Case(ii) If $N^* > \frac{K}{r}$ which is not possible.

Case (iii) If $N^* < \frac{K}{r}$ then $\frac{TrN^*}{K} e^{-\mu T} \sin \omega T < Te^{-\mu T} \sin \omega T < \frac{\pi}{2}$

$$\Rightarrow 0 < \mathrm{T} < \frac{\pi}{2}, \quad (32)$$

since $\sin \omega T < 1$ and $e^{-\mu t} \geq 1 \quad if \quad \mu \leq 0$

5 Conclusion

A model for growth of fish is given in section-II. A model for fishing economics giving the maximum sustainable yield is given in section-III. A delay model for fishing is given in section-IV and interpreted.

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