

A New Technique For Solving “Convex Hull” Problem

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Abstract

This paper presents a new technique for solving convex hull problem. To solve this problem the following steps have been used: i) Sorting the input points in ascending order (according to x-coordinate or y-coordinate). ii) Finding the leftmost and rightmost points in each horizontal line (HL) (when points are sorted according to y-coordinate) or top and bottom points in each vertical line (VL) (when points are sorted according to x-coordinate) including these sorted points. iii) Vertex recognition. If no. of HL or VL is k then recognizing $2k$ no. of points, the desired convex hull will be found although it contains kn no. of points (while each line has n points). Existing methodology computes the convex hull using all the input points. But in the new technique only $2k$ (where k is the no. of HL or VL) is used.

Keywords: Graham Scan method, Jarvis's March method, Divide and Conquer method, Incremental method, Prune- Search method.

Introduction

Convex hull is a part of computational geometry. Convex hull of a set S of points is the smallest convex polygon P for which each point in S is either on the boundary of P or in its interior. The convex hull of S is denoted by $CH(S)$. Convexity has a number of properties that make convex polygons easier to work with than arbitrary polygons. For example, every diagonal of a convex polygon is a chord, every vertex of convex polygon is convex (that means its interior angle is less than or equal to 180 degree). There are some methods have already generated for solving convex hull problem. Among these methods Graham Scan method^{1,5}, Jarvis's March method¹, Divide and Conquer method^{2,6-9}, Incremental method³ and Prune-Search methods⁴ are remarkable. Graham scan solves the convex hull problem by maintaining a stack Q of candidate points. Each point of the input set S is pushed once onto the stack, and the points that are not vertices of $CH(S)$ are eventually popped from the stack. When the algorithm terminates, stack Q contains exactly the vertices of $CH(S)$ in counter clockwise order of their appearance on the boundary. So the complexity in this method is increased with the increase of points. Jarvis's March computes the convex hull of a set S of points by a technique known as package wrapping (or gift wrapping). Jarvis's March simulates wrapping a taut piece of paper around the set S . By taping the end of the paper to the lowest point in the set is started, that is, to the same point p_0 with which Graham's scan is started. This point is a vertex of the convex hull. The paper is pulled to the right to make it taut, and then it is pulled higher until it touches a point. This point must also be a vertex of the convex hull. Keeping the paper taut, it is continued in this way around the set of vertices

until coming back to the original point p_0 . This method is asymptotically faster than Graham's scan. But its complexity is related with the number of points and no. of vertices. In Divide and Conquer method it divides total input points into two parts that means each part contains $kn/2$ (while $HL=k$ or $VL=k$ and each line has n points). Then generate two convex hulls and by combining these two hulls generate the desired $CH(S)$. Although the performance is good but its complexity is also depends on the no. input points. Prune and search method is similar to the worst case linear time algorithm. This method is asymptotically fastest. Its complexity depends on the no. of vertices and points. The complexity of Incremental method is also depends on no. of input points. So the complexity of all the method depends on mainly no. of input points or no. of vertices or both. But in the new method it depends on mainly number of lines.

Analysis of the method

This method is based on the following formulas:

- i) Every vertices of the convex hull must be the starting or ending points (among input points) of any horizontal or vertical line.
- ii) For top and bottom horizontal line or leftmost and rightmost vertical line the starting and ending points are must be vertices of desired convex hull.
- iii) The highest number of vertices in one line (horizontal or vertical) is less than or equal to two.
- iv) If the number of line (horizontal or vertical) is k at which all the points are lie then the total number of vertices is less than or equal to $2k$.

Methodology

Let a set S of n points. All these points are inserted into a smallest polygon such that the polygon is a convex polygon. To do so it is needed to sort the points in ascending order (according to x -coordinate or y -coordinate).

i) When the points are sorted according to y -ordinate then the leftmost point and rightmost point in each horizontal line are found out. If there is one point in any horizontal line then leftmost point is equal to rightmost point. Now two chains are created; one is left chain and another is right chain. If the no. of horizontal lines is k then each chain has k no. of points. Total no. of points $2k$ lie in two chains. Moreover it is said that vertices of the desired convex hull $CH(S)$ exist among these two chains.

Let left chain, $L = \{L_1, L_2, L_3, \dots, L_k\}$

And Right chain, $R = \{R_1, R_2, R_3, \dots, R_k\}$

where both L and R are the subsets of S .

By analysis, it is found L_1, R_1, R_k and L_k the vertices of $CH(S)$. Now it must be recognized the vertices from the rest points of two chains. For right chain it needs to find angles $\angle R_0 R_r R_i$ (where $R_0 = L_1$ and $R_r = R_1$. if $R_1 = L_1$ then $R_0 = R_1(x-1, y)$ and R_r is the recent vertex and $i = 2, 3, \dots, k$) and find the largest angle also. If $\angle R_0 R_1 R_5$ is largest then R_5 is the

recent vertex. Next step, it is essential to find the angle $\angle R_1 R_5 R_i$ (where $i = 6, 7, \dots, k$) and find the next vertex following the previous step. No need to find the angles for the points that are below the recent vertex R_5 because the upper point of the chain R_k is the vertex. So there is no probability to be the next vertex for points that are below the recent vertex. If for two points largest angle is same then farthest point is the vertex. Proceeding in this way it should be found out the vertices until arrived at R_k . For left chain it also be applied the same method. But in this case it will be started from L_k . For left chain it needs to find angles $\angle L_{k+1} L_k L_{k-i}$ (where $L_{k+1} = R_k$ if $L_k = R_k$ then $L_{k+1} = L_k(x+1, y)$ and L_k is the recent vertex and $i = 1, 2, 3, \dots, (k-1)$). All the vertices will be recognized from the left chain until arrived at L_1 .

For example: $n=31$ input points. $k=\text{No. of horizontal lines}=7$ which is shown in Figure1 and Figure2.

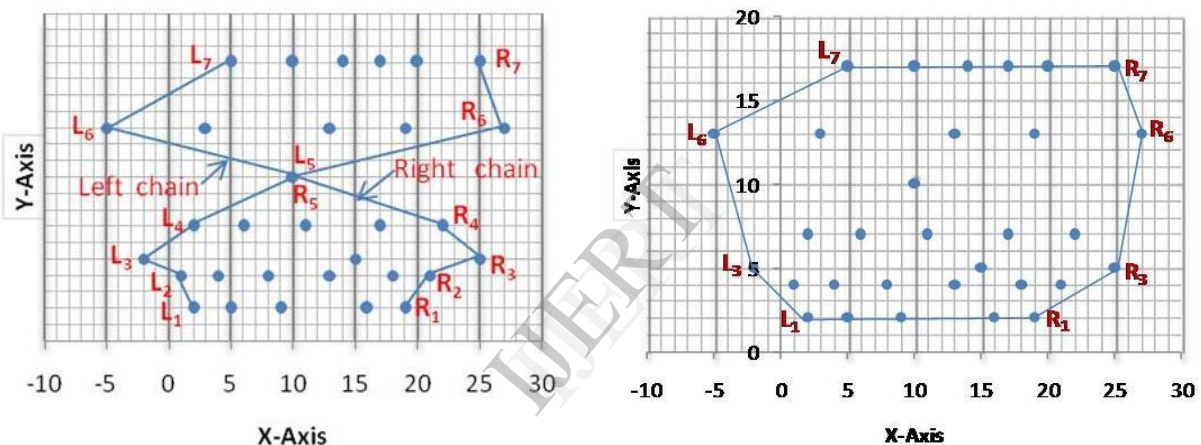


Figure-1: Two chains (when HL is consider) Figure-2: Convex hull (when HL is consider)

Here two chains: left chain, $L = \{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$

And right chain, $R = \{R_1, R_2, R_3, R_4, R_5, R_6, R_7\}$

In these two chains L_1, L_7, R_1 and R_7 are the known vertices. Between R_1 and R_7 the other vertices are R_3 and R_6 . To recognize these two vertices it needs to find the no. of angles as:

For $R_3 = \sum_{i=2}^7 \angle R_0 R_1 R_i$ (where $R_0 = L_1$) = 6. Here $\angle R_0 R_1 R_3$ is largest so R_3 is vertex.

For $R_6 = \sum_{i=4}^7 \angle R_1 R_3 R_i = 4$. Here $\angle R_1 R_3 R_6$ is largest so R_6 is the next vertex.

For sureness the next vertex $R_7 = 0$, since R_6 is vertex.

So total no. of angles in right chain = $6+4=10$.

Total no. of vertices in right chain $R = 2$ (without R_1 and R_7).

Again between L_1 and L_7 the other vertices are L_3 and L_6 . To recognize these two vertices it needs to find the no. of angles as:

For $L_6 = \sum_{i=1}^6 \angle L_8 L_7 L_{7-i}$ (where $L_8 = L_7$) = 6. where $\angle L_8 L_7 L_6$ is largest so L_6 is vertex.

For $L_3 = \sum_{i=1}^5 \angle L_7 L_6 L_i = 5$. where $\angle L_7 L_6 L_3$ is largest so L_3 is vertex.

For sureness the next vertex, $L_1 = \sum_{i=1}^2 \angle L_6 L_3 L_i = 2$. where $\angle L_6 L_3 L_1$ is largest angle so L_1 is the next vertex.

Total no. of angles in left chain $L = 6+5+2=13$

Total no. of vertices in left chain $L = 2$ (without L_1 and L_7).

The desired convex hull $CH(S) = R_1 R_3 R_6 R_7 L_7 L_6 L_3 L_1$.

Total no. of vertices in $CH(S) = 4+2+2=8$.

Total no. of angles to recognize 8 vertices is $13+10=23$.

ii) When the points are sorted according to x-ordinate then top and bottom point in each line are found out. In this case two chains (upper and lower) will be created. Where

Upper chain, $R = \{T_1, T_2, T_3, \dots, T_k\}$

Lower chain, $L = \{B_1, B_2, B_3, \dots, B_k\}$

L and R are both subset of S .

Here T_1 and T_k in R and B_1 and B_k in L are vertices known by sorting. Other vertices are recognized from the points of the two chains applying the same method that used in previous case.

For example, $n=34$ input points which are shown in figure-3 and figure-4.

No. of vertical lines=10. Here two chains,

Upper chain, $R = \{T_1, T_2, T_3, \dots, T_{10}\}$

Lower chain, $L = \{B_1, B_2, B_3, \dots, B_{10}\}$

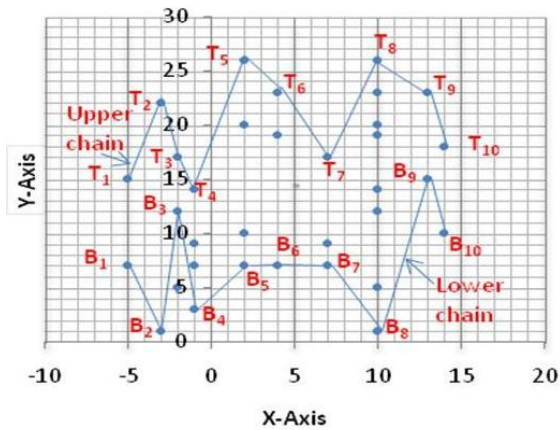


Figure-3: Two chains (when VL is consider)

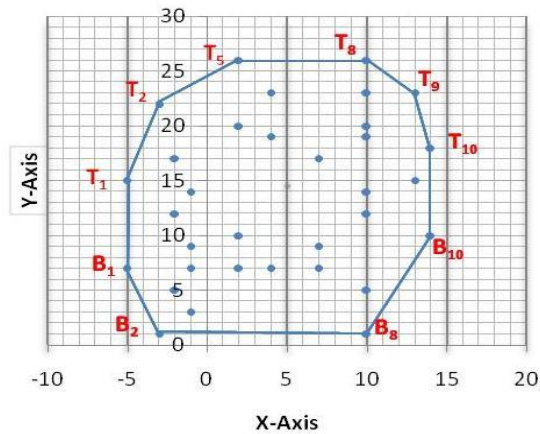


Figure-4: Convex hull (when HL is consider)

Among these two chains T_1 , T_{10} , B_1 and B_{10} are vertices known by sorting. The other vertices are T_2 , T_5 , T_8 , T_9 , B_2 , B_8 that are determined by applying the same method that used in previous case. So the convex hull $CH(S) = T_1 T_2 T_5 T_8 T_9 T_{10} B_{10} B_8 B_1 B_2$. Total no. of angles need to recognize all the vertices are $=28+19=47$.

Results and Discussion

From the above procedure it is seen that there are two major phases in the new method. One is sorting chain and another is vertices recognition phase. Vertex recognition phase is similar to Jarvis's March method. But there is a difference to recognize a vertex, it needs to find the angles with all the input point in Jarvis's March method where in the new method it needs to find the angles with the points that exist upper from recent vertex in that chain. Comparison between Jarvis's March method and the new method is shown in the following table.

No. of lines	Total no. of points	Input points	No. of vertices	Vertices of the CH(S)	No. of angles needed in Jarvis's March method	No. of angles needed in the new method
2	8	(2,1),(5,1),(9,1),(12,1), (-2,5), (2,5), (8,5), (10,5)	4	(2,1),(12,1), (-2,5), (10,5)	32	0
3	10	(-1,7), (4,7), (6,7), (10,7), (1,2),(5,2),(7,2),(-5,1), (1,1), (10,1)	4	(-5,1), (10,1), (-1,7), (10,7)	40	4
4	15	(-6,3),(-3,3),(0,3),(5,3), (2,5), (4,5), (10,5), (12,5), (15,5), (5,10) (7,10),(10,10),(15,10), (1,12), (5,12)	6	(-6,3),(5,3), (15,5),(15,10), (1,12),(5,12)	90	8
4	20	(-6,3),(-3,3),(0,3),(5,3), (10,3), (-10,5), (4,5), (10,5), (12,5), (15,5)(-10,10),(5,10), (7,10),(10,10),(15,10) (-2,12),(0,12),(1,12),(5,12), (7,12)	8	(-6,3), (10,3), (-10,5),(15,5), (-10,10), (15,10),(-2,12), (7,12)	160	9

5	15	(1,2),(4,2),(10,2), (-5,6), (5,6), (12,6), (1,10), (3,10), (5,10), (-3,15)(3,15),(4,15), (6,18),(10,18), (13,18)	7	(1,2),(10,2), (-5,6),(12,6), (-3,15),(6,18), (13,18)	105	13
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Table: Comparison between Jarvis's March method and the new method.

There is a similarity with the new method and Jarvis's March method so it is compared with Jarvis's March method and the method in the Table. From the Table it is seen that the no. of angles for recognizing the vertices depends on the number of lines and position of vertices on the chain, in the new method it does not depend on the no. of points. But in Jarvis's March method the no. of angles depends on both no. of vertices and the no. of points. From the above Table it is noticed that the performance of the new method is very high than J. March method.

Conclusion

This new method is a successful technique for solving convex hull problem. But there are some problems. In vertex recognition phase, when more than one angle is the largest then it is difficult to find out the farthest point. Moreover to find out the largest angle is also a problem. If it is considered the use orientation parameter for vertices recognition then its efficiency will be increased.

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