A Note On β-Content γ-Level Tolerance Interval For IFR Class Distribution

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ABSTRACT

Barlow and Proschan (1966) have obtained β-content γ-level Tolerance Intervals for IFR as well as IFRA class of distributions. Their results are based on inequalities for linear combinations of ordered statistics from IFR and IFRA families. The model considered in this chapter belongs to IFR class of distribution. We obtain parametric β-content γ-level Tolerance Interval for the same following Kumbhar and Shirke (2004). We further compare performance of parametric Tolerance interval with the one given by Barlow and Proschan (1966).

Key Words: β-content γ-level Tolerance Interval, IFR class of distribution.

1. Introduction

In general, term Tolerance Interval (TI) is an interval determined from observed values of a random sample for the purpose of drawing inferences about the proportion of a distribution contained in that interval. Usually TI is designed to capture at least a given proportion of some distribution. Two types of TI have received considerable attention in the literature; β-expectation TI and β-content γ-level TI. In order to be more specific about the meaning of TI, let X be a measurable characteristic having a distribution function $F(x;\theta)$, $\theta \in \Theta \subseteq \Re$. Let $L(\underline{X})$ and $U(\underline{X})$ be two functions of observations such that $L(\underline{X}) < U(\underline{X})$. Then $(L(\underline{X}), U(\underline{X}))$ is called a β−content γ−level TI, if for given $\beta, \gamma \in (0, 1)$,

$$P\{\int_{L(\underline{X})}^{U(\underline{X})} f(t;\theta)dt \geq \beta\} = \gamma, \qquad \text{for every } \theta \in \Theta, \qquad (1.1)$$

where $f(x;\theta)$ is probability density function (pdf) of X. The quantity $\int_{L(\underline{X})}^{U(\underline{X})} f(x;\theta)\,dx$ is called the sample coverage and $L(\underline{X})$ and $U(\underline{X})$ are called lower and upper tolerance limits, respectively. If we set $L(\underline{X}) = -\infty$ and obtain $U(\underline{X})$ satisfying (1.1) then we get upper β-content γ-level TI. Similarlly if we set $U(\underline{X}) = \infty$ and obtain L satisfying (1.1) then we get lower β-content γ-level TI. It is easy to observe that lower β-content γ-level tolerance limit is also an upper (1-β)-content (1-γ)-level tolerance limit. In the present study, we obtain only upper tolerance limits.

Wilks (1941) treated the problem of determining TIs in pioneer article. Since then a large number of papers dealing with this and other aspects of tolerance limits have appeared in the literature. Jilek (1981) classifies papers according to general results, distribution free results, normal and multivariate normal distributions, gamma, exponential, Weibull and other continuous and discrete distributions. Patel (1986) provided a review, which contains a large collection of known results on β–content γ–level TIs for some continuous and discrete univariate distributions.

Barlow and Proschan (1966) have obtained TIs for IFR as well as IFRA class of distributions. Their results are based on inequalities for linear combinations of ordered statistics from IFR and IFRA families. The model considered in this article belongs to IFR class of distribution (*Please see Lemma below*). Therefore, it is interesting to compare β–content γ–level TI based on MLE with respect to the one given by Barlow and Proschan (1966).

The Model

The cumulative distribution function (cdf) and probability density function (pdf) of lifetime distribution of largest of k independent and identically distributed (i.i.d.) exponential with mean θ are

$$F_X(x;\theta) = \begin{cases} \left(1-\exp(-x/\theta)\right)^k & x>0, \theta>0, \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

and

$$f_X(x;\theta) = \begin{cases} \frac{k}{\theta}\exp(-x/\theta)\left(1-\exp(-x/\theta)\right)^{k-1} & x>0, \theta>0, \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

respectively. The corresponding survival function is

$$S(x;\theta)=1-(1-\exp(-x/\theta))^k, \tag{1.4}$$

and the hazard function is

$$h(x;\theta) = \frac{k\exp(-x/\theta)(1-\exp(-x/\theta))^k}{\theta(1-(1-\exp(-x/\theta))^k}. \tag{1.5}$$

The graph of the harazad function is given in Fig.2.1. The following lemma proves that distribution defined in (1.2) is IFR.

Lemma 1.1: $F(t;\theta)=(1-\exp(-t/\theta))^k$ is a member of IFR class of distribution, for k>1.

Proof: Consider Harzard function defined as $h(t)=f(t;\theta)/\bar{F}(t;\theta)$,

where f(t; θ) is pdf as given in (2.4.2) and $\bar{F}(t;\theta)=1-F(t;\theta)$.

Therefore,

$$\log h(t) = \log\left(\frac{k}{\theta}\right) + (k-1)\log\left(1-\exp(-t/\theta)\right) - \frac{t}{\theta} - \log\left[1-\left(1-\exp(-t/\theta)\right)^k\right] \tag{1.6}$$

If $y=\left(1-\exp(-t/\theta)\right)$, equation (1.6) reduces to

$$\log h(t) = \log h(y) = c + (k-1)\log y - \log(1-y) - \log(1-y^k).$$

Differentiating with respect to y we get

$$\frac{d\log h(y)}{dy} = \frac{(k-1)}{y} + \frac{1}{(1-y)} + \frac{ky^{k-1}}{(1-y^k)}. \tag{1.7}$$

When k=1, it corresponds to exponential distribution. For k ≥2, (k-1) ≥ 1 and since

$0 < y < 1$; $\frac{d\log h(y)}{dt} > 0 \quad \forall \quad k \geq 2$. Hence the Lemma. □

We further note that as long as $k \geq 1$ (even not integer) IFR property is retained for this distribution.

2. β-content γ-level TI for lifetime distribution of k-unit parallel system

Let $I(\underline{X}) = (0\ ,\ \delta\hat{\theta})$ be an upper β–content γ–level TI for the distribution having distribution function (1.2). The constant $\delta\ (>0)$ for $\beta \in (0,1)$, $\gamma \in (0,1)$ is to be determined such that

$$P\{F(\delta\hat{\theta}\,;\,\theta) \geq \beta\} = \gamma. \tag{2.1}$$

Using asymptotic normality of $\hat{\theta}$, (2.1) can be equivalently written as

$$P\ \{Z \leq -(\theta / \sigma(\theta))\,[1 + \log(1-\beta^{1/k}) / \delta]\} = 1-\gamma,$$

where $Z \sim N(0, 1)$. This gives

$$\delta = -\log(1-\beta^{1/k}) / [1 + Z_{1-\gamma}\,\sigma(\theta)/\theta],$$

where $Z_{1-\gamma}$ is the $100(1-\gamma)^{th}$ lower percentile of the standard normal distribution. Define

$$A_k = k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(j+1)^2} - k(k-1)\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(j+2)^2} + k(k-1)\sum_{j=0}^{k-3}\binom{k-3}{j}\frac{(-1)^j}{(j+2)^3},$$

where ${}^{a}C_{b} = a!/(b!(a-b)!)$.

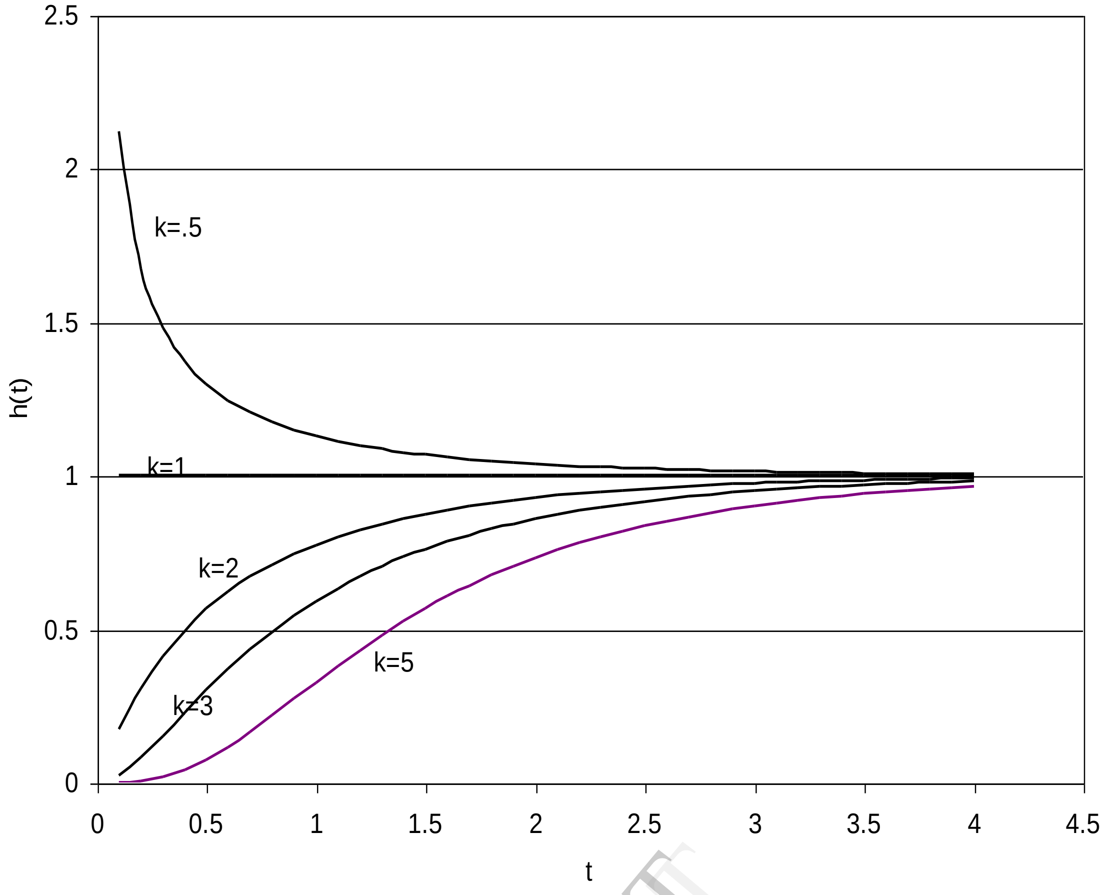
Then an upper tolerance limit of β - content γ - level TI, $I_2(X)$ is given by,

$$U(\underline{X}) = \hat{\theta}\ \{-\log(1-\beta^{1/k}) / [1 + Z_{1-\gamma}\,\sigma(\theta)/\theta]\}, \tag{2.2}$$

where

$$\sigma^2(\theta) = \begin{cases} \theta^2/n & k=1 \\ 0.55310\theta^2/n & k=2\,; \\ (2A_k-1)^{-1}\theta^2/n & k\geq 3 \end{cases} \tag{2.3}$$

Figure 2.1 : Hazard Function of F(.,.)



It is clear that $\sigma(\theta)/\theta$ in (2.5) is free from θ for all $k \geq 1$. Furthermore, $I(\underline{X})$ is not an exact β–content γ–level upper TI, since the constant δ is determined using asymptotic normality. This necessitates study of confidence level γ of $I(\underline{X})$ for various values of n, β and θ. [Details of performance of (2.2) is reported in Kumbhar and Shirke (2004).]

IFR Class Tolerance Interval

Upper TI for $F \in$ IFR class of distributions following Barlow and Proschan (1966) is as follows:

$$I_{BP}(\underline{X}) = (0, \; C_{\beta,\gamma,n}\, \hat{\theta}_n)\,; \qquad (2.3)$$

where $$C_{\beta,\gamma,n} = \frac{-2n\log(1-\beta)}{\chi^2_{2n,\gamma}}$$

with $\chi^2_{2n,\gamma}$ is the upper γ-percent percentile of the chi-square distribution with 2n degrees of freedom and

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}(n-i+1)\left(X_{(i)} - X_{(i-1)}\right); \qquad (2.4)$$

with $X_{(i)}$ is the i^{th} ordered statistic in the sample $X_1, X_2, \ldots, X_n$. Thus for the model (2.1) we have two TIs. Therefore it is interesting to see relative performance of these TIs.

3. Comparison of MLE based and IFR class Tolerance intervals

In order to compare the two TIs, we conduct simulation experiment for various values of n, β, γ and θ =1 and 2, when k=2. We generate 25000 samples of size n each from (1.2). Upper TLs of $I_{BP}(\underline{X})$ and $I(\underline{X})$ are computed for each sample and the average upper TLs of the same are taken. Please refer to Table 3.1 and Table 3.2 for the simulated average upper TLs. We have also conducted simulation experiment to observe the effect of number of units in the parallel system on width of both the above TIs. The results are tabulated in Table 3.3.

Table 3.1:Comparison of MLE based and IFR class Upper TLs for θ =1 and k=2.

Sample Size (n)	γ =0.90					
	β=0.90			β=0.95		
	$U_{MLE}(\underline{X})$	$U_{BP}(\underline{X})$	Difference	$U_{MLE}(\underline{X})$	$U_{BP}(\underline{X})$	Difference
10	4.278070	5.189807	0.911737	5.280729	6.734745	1.454016
20	3.779746	4.594466	0.814720	4.681151	5.980370	1.299219
25	3.671747	4.455863	0.784116	4.548275	5.803499	1.255224
50	3.434548	4.138115	0.703537	4.252196	5.384721	1.132525
γ =0.95						
10	4.859227	5.933962	1.074735	6.015581	7.724343	1.708762
20	4.085189	5.027972	0.942783	5.065436	6.552964	1.487528
25	3.928513	4.828962	0.900449	4.867146	6.281099	1.413953
50	3.59487	4.376463	0.781593	4.442412	5.685691	1.243279

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Table 3.2:Comparison of MLE based and IFR class Upper TLs for θ=2 and k =2.

Sample Size (n)	γ =0.90					
	β=0.90			β=0.95		
	$U_{MLE}(\underline{X})$	$U_{BP}(\underline{X})$	Difference	$U_{MLE}(\underline{X})$	$U_{BP}(\underline{X})$	Difference
10	8.539951	10.353682	1.813731	10.559595	13.452091	2.892496
20	7.568627	9.201962	1.633335	9.363304	11.967651	2.604347
25	7.331238	8.899210	1.567972	9.104967	11.616709	2.511742
50	6.858973	8.263653	1.40468	8.510165	10.776218	2.266053
γ =0.95						
10	9.694414	11.843263	2.148849	12.009997	15.429117	3.41912
20	8.183687	10.072673	1.888986	10.098490	13.067099	2.968609
25	7.863148	9.663821	1.800673	9.735890	12.561660	2.82577
50	7.172979	8.732775	1.559796	8.893104	11.381074	2.48797

Table 3.3: Comparison of MLE based and IFR class Upper TLs when θ =1, β = 0.90 and γ = 0.90 .

Sample Size (n)	Number of units (k)	$U_{MLE}(\underline{X})$	$U_{BP}(\underline{X})$	Difference
10	2	4.263407	5.547987	1.284581
	3	4.546516	6.774717	2.228201
	5	4.927414	8.428308	3.500894
20	2	3.778254	4.750770	0.972516
	3	4.121627	5.813017	1.691390
	5	4.559469	7.228765	2.669296
25	2	3.670547	4.577007	0.906461
	3	4.023212	5.597181	1.573968
	5	4.469737	6.957966	2.488229
50	2	3.434350	4.192794	0.758436
	3	3.810167	5.131015	3.320848
	5	4.279542	6.382337	2.102795

4. Conclusions

From the Table 3.1 to Table 3.3 we observe that

i) An average upper TL based on MLE is below the average upper TL based on Barlow and Proschan (1966) procedure for all combinations of n, β, γ and θ. The difference in length increases as θ increases.

ii) As n increases the difference between these average TLs does not reduce rapidly.

iii) Increase in number of units in the parallel system; k has significant effect on the length of TI due to Barlow and Proschan (1966). The length of $I_{BP}(\underline{X})$ increases rapidly as compared to the length of $I(\underline{X})$. as k increases the hazard function of F(., θ) moves away from hazard function of exponential distribution.[Please see Fig.2.1).]

Therefore, while using $I_{BP}(\underline{X})$ it should be kept in mind that, length of TI has significant effect as distribution moves away from exponentiality. It should be interesting to see performance of $I_{BP}(\underline{X})$ with respect to hazard rate ingeneral. One of the reasons for this observation about $I_{BP}(\underline{X})$ could be use of stochastic order between exponential distribution and distribution belonging to IFR class. Therefore as distribution moves away from exponential distribution with respect to hazard rate, TI will havs significant effect on the length.

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