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A Two Dimensional Infection Age – Structured Mathematical Model Of The Dynamics Of HIV/AIDS.

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Abstract

The paper considers two compartmentalized mathematical model of HIV/AIDS disease dynamics of the susceptible and infected members with age structure introduced in the latter class. The susceptables are virus free but prone to infection through specific transmission pattern that is, coming in contact with infected body fluids such as blood, sexual fluids and breast milk. The total population is partitioned into two distinct classes giving rise to a set of model equations with one ordinary differential equation and one partial differential equation. Parameter values were used to represent the consequential interactive characteristics of the population. The equilibrium states and the corresponding characteristics equation were obtained. The Bellman and Cooke's theorem is applied to analysed the equilibrium states of the model for stability and critical values of these parameters obtained. The result revealed that to sustain the population, the birth rate must be greater than the death rate among others.

1. Introduction

According to Benyah [3], mathematical modeling is an evolving process, as new insight is gained the process begins again as additional factors are considered. The research work proposes a deterministic mathematical model of HIV/AIDS disease dynamics resulting into a system of ordinary and partial differential equations. Differential equations form very important mathematical tools used in producing models of physical and biological processes. Burghes and Wood [4] opines that "…it could even be claimed that the spread of modern industrial civilization, for better or for worse, is partly a result of man's ability to solve the differential equations which govern so many of our industrial processes, be them chemical or engineering".

In this work, the population is partitioned into two compartments of the susceptible S(t), which is the class of members that are virus – free but are prone to infection as they interact with the infected class.

The infected class I(t) consists of members that contracted the virus and are at various stages of infection. This class is structured by the infection age, with the density function ρ(t, a) where 't' is the time and 'a' is the infection age.

There is a maximum infection age T at which a member of the infected class must leave the compartment via death i.e. when a = T for $0 \leq a \leq T$. However, a member of the class could still die by natural causes at a rate μ, which is also applicable to the susceptible class S(t).

Members of S(t) move into I(t) at a rate α due to negative change in behavior. The gross death rate via infection is given by $\sigma(a) = \mu + \delta \tan \frac{\pi a}{2TK}$, δ is additional burden from infection while K is a control parameter associated with the measure of slowing down the death of the infected member, such as the effectiveness of the anti – retroviral drugs which give the victims longer life – span. A high rate of 'K' will imply high effectiveness of such measure and vice – versa.

It is assumed that while the new births in S(t) are born there in, the off-springs of I(t) are divided between S(t) and I(t) in the proportion θ and $1 - \theta$ respectively that is, $1 - \theta$ of the off-springs of I(t) are born with the virus.

2. The model equations

$S^1 = (\beta - \mu)s(t) + \theta\beta I(t) - \propto s(t)I(t)$ (1)

and $I(t) = \int_0^T \rho(t, a)da,\ 0 \le a \le T$ (2)

$\frac{\partial \rho(t,a)}{\partial t} + \frac{\partial \rho(t,a)}{\partial a} + (t + \sigma(a))\rho(t, a) = 0$ (3)

Where $\sigma(a) = \mu + \delta \tan \frac{\pi a}{2TK}$ (4)

$\rho(t, 0) = B(t) = \propto s(t)I(t) + (1-\theta)\beta I(t)$ (5)

and $\rho(0, a) = \varphi(a)$ (6)

$S(0) = S_0, I(0) = I_0$ (7)

With the parameters given by

$\beta =$ natural birth rate for the population;

$\mu =$ natural death rate for the population.

$\propto =$ rate of contracting the HIV virus.

$\sigma(a) =$ gross death rate of the infected class.

$\delta =$ additional burden from infection.

K = measure of the effectiveness of efforts at slowing down the death of infected members.

$\theta =$ the proportion of the off-springs of the infected which are virus free at birth $0 \le \theta \le 1$.

T = maximum infection age i.e. when a = T the infected member dies of the disease.

3. Equilibrium states

At the equilibrium states, let

$S(0) = x,\ I(0) = y$ (8)

if $\rho(t, a) = \varphi(a)$ (9)

from (1.2), $y = \int_0^T \varphi(a)$ (10)

from (5), $\varphi(0) = \beta(0) = \propto xy + (1-\theta)\beta y$ (11)

Substituting (9) to (11) into (1) and (3)

$(\beta - \mu)x + \theta\beta y - \propto xy = 0$ (12)

$\frac{d\varphi\varnothing(a)}{da} + \sigma(a)\varphi(a) = 0$ (13)

$\frac{d\varnothing\varphi(a)}{\varphi(a)} = -\sigma(a)da$ (14)

Integrating both sides

$\varphi(a) = \varphi(0) \exp\{-\int_0^a \sigma(s)\,ds\}$ (15)

Let $\pi(a) = \exp\{-\int_0^a \sigma(s)ds\}$ (16)

That is,

$\varphi(a) = \varphi(0)\pi(a)$ (17)

and

$y = \varphi(0)\int_0^T \pi(a)da = \varphi(0)\overline{\pi}$ (18)

Using (11) and (18)

$y = (\propto xy + (1-\theta)\beta y)\overline{\pi}$ (19)

From (12) and (19).

$x = \frac{1}{\propto}(1 - (1-\theta)\beta\overline{\pi})$ (20)

Substituting (20) in (12)

$y = \frac{(\beta-\mu)\frac{1}{\propto}(1-(1-\theta)\beta\overline{\pi})}{[(1-(1-\theta)\beta\overline{\pi}) - \theta\beta]}$ (21)

Hence, the zero equilibrium state, is $(x, y) = (0,0)$ and the non-zero equilibrium state is given by (20) and (21).

4. The characteristics equation

As in Akinwande [1], let the equilibrium state be perturbed as follows:

$S(t) = x + p(t),\ p(t) = \overline{p}e^{\lambda t}$ (22)

$I(t) = y + q(t);\ q(t) = \overline{q}e^{\lambda t}$ (23)

Let $\rho(t. a) = \varphi(a) + \eta(a)e^{\lambda t}$ (24)

With $\overline{q} = \int_0^T (a)da$ (25)

Substituting (22) to (25) into the model equations (1) and (3)

$\lambda) d\frac{d}{dt}(x + \bar{p}e^{\lambda t}) = (\beta-\mu)\ (x+\bar{p}e^{\lambda t}) + \theta\beta(y+\bar{q}e^{\lambda t}) -\propto (x+\bar{p}e^{\lambda t})(y+\bar{q}e^{\lambda t})$

$\lambda\bar{p}e^{\lambda t} = (\beta-\mu)\ x + (\beta-\mu)\bar{p}e^{\lambda t} + \theta\beta y + \theta\beta\bar{q}e^{\lambda t} -\propto xy - \propto x\bar{q}e^{\lambda t} -\propto y\bar{p}e^{\lambda t} - \propto\bar{p}\bar{q}e^{2\lambda t}$

From equation (12) and neglecting terms of order 2, we have;

$\lambda\bar{p}e^{\lambda t} = (\beta-\mu)\bar{p}e^{\lambda t} + \theta\beta\bar{q}e^{\lambda t} -\propto x\bar{q}e^{\lambda t} -\propto y\bar{q}e^{\lambda t}$

or $(\beta-\mu-\propto y-\lambda)\bar{p} + (\theta\beta-\propto x)\bar{q} = 0$ (26)

Substituting (24) into (3), gives

$\frac{d\rho}{dt}(\varphi(a)\ \eta(a)\ e^{\lambda t}] + \frac{d}{da}[\varphi(a)+\eta(a)\ e^{\lambda t}] + \sigma(a)[\varphi(a)+\eta(a)e^{\lambda t}] = 0$

That is,

$\lambda\eta(a)e^{\lambda t} + \frac{d\varphi(a)}{da} + e^{\lambda t}\frac{d}{da}\eta(a) + \sigma(a)\varphi(a) + \sigma(a)\eta(a)e^{\lambda t} = 0$

Since

$\frac{d\varphi(a)}{da} + \sigma(a)\varphi(a) = 0$

Then

$\lambda(a)e^{\lambda t} + e^{\lambda t}\frac{d}{da}\eta(a) + \sigma(a)\eta(a)e^{\lambda t} = 0$

$\frac{d}{da}\eta(a) + (\sigma(a) + \lambda)\ \eta(a) = 0$ (27)

Solving the Ordinary Differented Equation (27), gives

$$\frac{d\eta(a)}{\eta(a)} = -(\sigma(a) + \lambda)da \qquad (28)$$

$\eta(a) = \eta(0) \exp\{-\int_0^a(\sigma(s) + \lambda)ds\}$ (29)

Integrating (29) over [0, T] gives

$\bar{q} = \eta(0)\int_0^T[\exp\{-\int_0^a(\sigma(s) + \lambda)ds\}]\,da$

or $\bar{q} = \eta(0)\ b(\lambda)$ (30)

Since $\bar{q} = \eta(0)\ b(\lambda)$,where $b(\lambda) = \int_0^T \exp\{-\int_0^a(\sigma(s) + s\}]\,da$ (31)

$\eta(0)$ is calculated as follows:

From (11), $\varphi(0) = \propto xy + (1-\theta)\beta y$.

and (24) $\rho(t, a) = \varphi(a) + \eta(a)\ e^{\lambda t}$

But $\rho(t, 0) = \beta(t) = \varphi(0) + \eta(0)\ e^{\lambda t}$ (32)

From (5), $\beta(t) = \propto s(t) + (1-\theta)\beta I(t)$

Substituting (22) to (25) into (5) and using (11) and (32)

$B(t) = \propto(x + \bar{p}e^{\lambda t})(y + \bar{q}e^{\lambda t}) + (1-\theta)\beta\ (y + \bar{q}e^{\lambda t})$

$= \propto xy + \propto\bar{p}ye^{\lambda t} + \propto x\bar{q}e^{\lambda t} + \propto\bar{p}\bar{q}e^{2\lambda t} + (1-\theta)\beta y + (1-\theta)\beta\bar{q}e^{\lambda t}$ (33)

Compare this with (32) using (11) for $\varphi(0)$ gives

$\propto xy + (1-\theta)\beta y + \eta(0)e^{\lambda t} = \propto xy + \propto\bar{p}ye^{\lambda t} + \propto x\bar{q}e^{\lambda t} + \propto\bar{p}\bar{q}e^{2\lambda t} + (1-\theta)\beta y + (1-\theta)\beta\bar{q}e^{\lambda t}$

neglecting terms of order 2

$$\eta(0) = \propto\bar{p}y + \propto\bar{q}\chi + (1-\theta)\beta\bar{q} \qquad (34)$$

Substituting $\eta(0)$ in (30)

$$\bar{q} = (\propto y\bar{p} + \propto x\bar{q} + (1-\theta)\beta\bar{q})\ b(\lambda) \qquad (35)$$

$$\propto y\bar{p} + [(\propto x + (1-\theta)\beta)\ b(\lambda)-1)]\ \bar{q} = 0 \qquad (36)$$

Using (26) and (36), we obtain the Jacobian determinant for the system with the eigen value λ

$$\begin{vmatrix} \beta-\mu-\propto y-\lambda & \theta\beta-\propto x \\ \propto y & (\propto x + (1-\theta)\beta)b(\lambda)-1 \end{vmatrix} = 0 \qquad (37)$$

and the characteristics equation is given by:

$(\beta-\mu-\propto y-\lambda)\ [(\propto x + (1-\theta)\beta)b(\lambda)-1] -\propto y(\theta\beta-\propto x) = 0$ (38)

5. Stability of the zero equilibrium state

At the zero equilibrium state (x, y) = (0, 0), the characteristic equation becomes:

$$(\beta - \mu - \lambda)[(1-\theta)\beta b(\lambda) - 1] = 0 \quad (39)$$

That is,

either $(\beta - \mu - \lambda) = 0$ or $(1-\theta)\beta b(\lambda) - 1 = 0$ (40)

$$\lambda_1 = \beta - \mu \quad (41)$$

This shows that $\lambda_1 < 0$, if $\beta < \mu$

The nature of the roots of the transcendental equation $(1-\theta)\beta b(\lambda) - 1$ is now investigated.

Since $b(\lambda) = \int_0^T \exp\{-\int_0^a (\lambda + \sigma(s)\}\, da$, which implies that

$$b(\lambda) = \int_0^T e^{-\lambda a}\pi(a)\, da \quad (42)$$

Using the approximation

$$b(\lambda) = \int_0^T (1-\lambda a)\ \pi(a)\ da = \int_0^T \pi(a)\, da - \lambda \int_0^T a\pi(a)\, da \quad (43)$$

$$= \bar{\pi} - \lambda A \text{ where } A = \int_0^T a\pi(a)\, da$$

So, $(1-\theta)\beta b(\lambda) - 1 = 0$ takes the form:

$$(1-\theta)\beta(\bar{\pi} - \lambda A) - 1 = 0 \quad (44)$$

$$\lambda = \frac{(1-\theta)\beta\bar{\pi} - 1}{(1-\theta)\beta A} \quad (45)$$

So, sign λ = sign $\{(1-\theta)\beta\bar{\pi} - 1\}$ (46)

Let $D_1(k) = (1-\theta)\beta\bar{\pi} - 1$ (47)

So, the origin will be stable when $D_1(k) < 0$ and Unstable when otherwise.

Table 1. Stability analysis of the zero state.

K	δ	θ	T	β	μ	D_1(k)	Remark
0.3	0.003	0.4	10	0.31	0.12	0.0008826	Instability
0.4	0.003	0.4	10	0.31	0.12	0.0014138	Instability
0.5	0.003	0.4	10	0.31	0.12	0.0011976	Instability
0.6	0.003	0.4	10	0.31	0.12	0.0010943	Instability
0.7	0.003	0.4	10	0.31	0.12	0.0009917	Instability
0.8	0.003	0.4	10	0.31	0.12	0.0008181	Instability
0.9	0.003	0.4	10	0.31	0.12	0.0002778	Instability

From table 1, $D_1(k) > 0$ when $\beta > \mu$, which implies the instability of the origin.

6. Stability analysis of the non zero state

At the non zero state

$$(x, y) = \left\{\frac{1}{\propto}(1 - (1-\theta)\beta\bar{\pi}, \frac{(\beta-\mu)\frac{1}{\propto}[1-(1-\theta)\beta\bar{\pi}]}{\{[1-(1-\theta)\beta\bar{\pi}] - \theta\beta\}}\right\} \quad (48)$$

To analyse the non – zero state for stability, we shall apply the result of Bellman and Cooke [2] to the characteristics equation (38) taking it in the form $H(\lambda) = 0$

if we set $\lambda = iw$; and have $H(iw) = F(w) + iG(w)$ (49)

Since $b(\lambda) = \int_0^T \exp\{-\int_0^a (\lambda + \sigma(s)ds\}\, da$

$$b(iw) = \int_0^T \exp\{-\int_0^a (iw + \sigma(s)\}\, da = \int_0^a e^{-iwa}\bar{\pi}(a)da \quad (50)$$

$$= \int_0^T [(\cos wa - (i\sin wa)]\ \pi(a)\, da = f(w) + ig(w) \quad (51)$$

in $f(w) = \int_0^T \pi(a)\cos wa\, da$ (52)

and $g(w) = -\int_0^T \pi(a)\sin wa\, da$ (53)

so , $f(0) = \int_0^T \pi(a)da = \bar{\pi}$ (54)

$g(0) = 0$ (55)

Also , $f(w) = -\int_0^T a\pi(a) \sin(wa)\, da$

$f(0) = 0$ (56)

and, $g(w) = -\int_0^T a\pi(a) \cos wa\, da$

$g(0) = -\int_0^T a\pi(a)da = -A$ (57)

Thus, $H(iw) = (\beta - \mu - \propto y - iw)\,[(\propto x + (1-\theta)\beta)\, b(iw) - \propto y\,(\theta\beta - \propto x)$ (58)

$H(iw) = (\beta - \mu - \propto y - iw)\,[(\propto x + (1-\theta)\beta)\, f(w) + ig(w) - 1] - \propto y\,(\theta\beta - \propto x)$

$= (\beta - \mu - \propto y - iw)\,[(\propto x + (1-\theta)\beta)\, f(w) + (\propto x + (1-\theta)\beta) ig(w) - 1] - \propto y\,(\theta\beta - \propto x)$

$F(w) = (\beta - \mu - \propto y)\,(\propto x + (1-\theta)\beta)\, f(w) + w(\propto x + 1-\theta)\beta) g(w) - (\beta - \mu - \propto y) - \propto y\,(\theta\beta - \propto x)$ (59)

$G(w) = (\beta - \mu - \propto y)\,(\propto x + (1-\theta)\beta)\, g(w) - w\,(\propto x + (1-\theta)\beta)\, f(w) + w$ (60)

$G(0) = 0$

$F(0) = (\beta - \mu - \propto y)\,(\propto x + (1-\theta)\beta)\, \bar{\pi} - (\beta - \mu - \propto y) - \propto y(\theta\beta - \propto x)$ (61) and

$G^1(0) = -(\beta - \mu - \propto y)\,(\propto x + (1-\theta)\beta)A - (\propto x + (1-\theta)\beta)\, \bar{\pi} + 1$ (62)

$F^1(0) = 0$

For stability or otherwise of the equilibrium state, we need to satisfy the condition for which the inequality $F(0)\, G^1(0) - F^1(0)\, G(0) > 0$ holds.

The inequality then gives $F(0)\, G^1(0) > 0$

Let $D_2(k) = F(0)\, G^1(0)$

Then, the non-zero state will be stable when $D_2(k) > 0$ (63)

Table 2. Stability analysis of the non-zero state.

K	β	δ	μ	∝	θ	T	$D_2(k)$	Remark
0.1	0.2	0.01	0.015	0.001	0.4	1	0.02328541	Stable
0.2	0.2	0.01	0.015	0.002	0.4	1	0.02325401	Stable
0.3	0.2	0.01	0.015	0.003	0.4	1	0.02355326	Stable
0.4	0.2	0.01	0.015	0.004	0.4	1	0.02368585	Stable
0.5	0.2	0.01	0.015	0.005	0.4	1	0.02381756	Stable
0.6	0.2	0.01	0.015	0.006	0.4		0.02394839	Stable
0.7	0.2	0.01	0.015	0.007	0.4	1	0.02407834	Stable
0.8	0.2	0.01	0.015	0.008	0.4	1 1	0.02420741	Stable
0.9	0.2	0.01	0.015	0.009	0.4		0.02433560	Stable
1	0.2	0.01	0.015	0.01	0.4	1 1	0.02446291	Stable

From table 2, $D_2(k)>0$ which implies the stability of the non zero state.

Conclusion

The zero equilibrium state which is the state of population extinction will be stable when the birth rate is less than the death rate in addition to meeting the requirement of inequality $D_1(k)<0$. The non zero state, which is the state of population sustenance will be stable if the inequality (63) is satisfied. So efforts must be geared toward meeting this non zero stability bound through public enlightenment.

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