Abstract Harmonic Analysis and Ideals of Banach Algebra on 3-Step Nilpotent Lie Groups.

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Abstract

Let N be the $6-$dimensional nilpotent Lie group and let $\mathbb{R}^9$ be its vector group. we construct a $9-$dimensional new group that contains the two groups N and $\mathbb{R}^9$. We will define the Fourier transform on N, in order to obtain the Plancherel theorem. Moreover, we show how F. Treves and M. Atiyah methods can be used to obtain the division of distributions on N. To this end, a classification of all ideals of the Banach algebra $L^1(N)$ of N will be obtained.

Keywords: $3-$Step Nilpotent Lie Groups, Plancherel Formula, Partial Differential Equations, Ideals of Banach Algebra $L^1(N)$.

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1 Introduction.

1.1. The abstract harmonic analysis, is a powerful area of pure mathematics that has connections to analysis, algebra, geometry, theoretical physics and solving problems in robotics, image analysis, mechanics,engineering. Abstract harmonic analysis is the field in which results from Fourier analysis are extended to topological groups which are not commutativ This analysis is generally a hard theory and its difficulty makes the noncommutative version of the problem very challenging. The main task is therefore the case of Lie groups which is locally compact, not compact and not commutative. If the Lie group N is assumed to be noncommutative, it is not possible anymore to consider the dual group $\widehat{N}$. Recently, this problem found a satisfactory solution with my papers $[4, 5, 7]$.

Here are some interesting examples of these groups. Let G_5 be the real group consisting of all matrices of the form

$$\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & -x_1 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & \frac{x_2^2}{2} & x_5 - \frac{x_1 x_2^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & x_2 & -x_3 - x_1 x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

G_5 (is called the Cartan group G_5 or the generalized Dido problem), where $(x_5, x_4, x_3, x_2, x_1) \in \mathbb{R}^5$. Let $K = \mathbb{R}^5$ be the group with law

$$\begin{aligned} & (x_5, x_4, x_3, x_2, x_1)(y_5, y_4, y_3, y_2, y_1) \\ = \quad & (x_5 + y_5 + \frac{1}{2}x_1 y_2^2 - x_2 y_3 + x_1 x_2 y_2, x_4 + y_4 + \frac{1}{2}x_1^2 y_2 - x_1 y_3, y_3 + x_3 - x_1 y_2, x_2 + y_2, x_1 + y_1) \end{aligned}$$

for any $(x_5, x_4, x_3, x_2, x_1) \in \mathbb{R}^5$ and $(y_5, y_4, y_3, y_2, y_1) \in \mathbb{R}^5$. The inverse of an element $(x_5, x_4, x_3, x_2, x_1)$ is

$$\begin{aligned} & (x_5, x_4, x_3, x_2, x_1)^{-1} \\ = \quad & (-x_5 - \frac{x_1}{2}x_2^2 - x_2 x_3, -x_4 - \frac{x_1^2}{2}x_2 - x_1 x_3, -x_3 - x_1 x_2, -x_2, -x_1) \end{aligned} \tag{2}$$

Dixmier had proved in $[2, p.331]$ that there is a group isomorphism between G_5 and K. Thanks to this isomorphism, the group K can be shown as a semi-direct product $\mathbb{R}^3 \underset{\rho_2}{\ltimes} \mathbb{R} \underset{\rho_1}{\ltimes} \mathbb{R}$ of the real vector groups $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^3$ see [4], where ρ_2 is the group homomorphism $\rho_2 : \mathbb{R} \to Aut(\mathbb{R}^3)$, which is defined by

$$\rho_2(x_2)(y_5, y_4, y_3) = (y_5 - x_2 y_3, y_4, y_3) \tag{3}$$

and ρ_1 is the group homomorphism $\rho_1 : \mathbb{R} \to Aut(\mathbb{R}^3 \underset{\rho_2}{\ltimes} \mathbb{R})$, which is given by

$$\rho_1(x_1)(y_5, y_4, y_3, y_2) = (y_5 + \frac{x_1}{2}y_2^2, y_4 + \frac{x_1^2}{2}y_2 - x_1 y_3, y_3 - x_1 y_2, y_2) \tag{4}$$

where $Aut(\mathbb{R}^3)$ $(resp. Aut(\mathbb{R}^3 \underset{\rho_2}{\ltimes} \mathbb{R}))$ is the group of all automorphisms of $(\mathbb{R}^3)$ $(resp. (\mathbb{R}^3 \underset{\rho_2}{\ltimes} \mathbb{R}))$. Let N be the real group consisting of all matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_3 & x_6 \\ 0 & 1 & x_2 & x_5 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

where $(x_6, x_5, x_4, x_3, x_2, x_1) \in \mathbb{R}^6$. The group can be identified with the group $(\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R}^2\ ) \underset{\rho_1}{\rtimes} \mathbb{R}$ be the semidirect product of the real vector groups $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, where ρ_2 is the group homomorphism $\rho_2 : \mathbb{R}^2 \to Aut(\mathbb{R}^3)$, which is defined by

$$\rho_2(x_3, x_2\ )(y_6, y_5, y_4) = (y_6 + x_3y_4, y_5 + x_2y_4, y_4) \tag{6}$$

and ρ_1 is the group homomorphism $\rho_1 : \mathbb{R} \to Aut(\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R}^2)$, which is given by

$$\rho_1(x_1\ )(y_6, y_5, y_4, y_3, y_2) = (y_6 + x_1y_5, y_5, y_4, y_3 + x_1y_2, y_2) \tag{7}$$

where $Aut(\mathbb{R}^3)$ $(resp.Aut(\mathbb{R}^3 \underset{\rho_2}{\rtimes} \mathbb{R}^2))$ is the group of all automorphisms of $(\mathbb{R}^3)$ $(resp.(\mathbb{R}^3 \underset{\rho_2}{\ltimes} \mathbb{R}^2))$, see [4]. and the inverse of an element

$$(x_6, x_5, x_4, x_3, x_2, x_1) = (-x_6-x_3x_4-x_1x_5-x_1x_2x_4, -x_5-x_2x_4, -x_4, -x_3-x_1x_2, -x_2, -x_1) \tag{8}$$

1.2. Using the technique in $[4, 6, 7]$ as a guideline, our goal in this paper is to generalize the classical Fourier transform on $3-$ step nilpotent Lie groups, to obtain the following results

I- Plancherel formula on these groups theorem 2.1

II- Theorems 3.1 *and* 3.2 *for the division of distributions, and classification of all left ideals of Banach algebra* $L^1(N)$ *theorem* 4.1 *and corollary* 4.2

The point I wish to make in this paper that the Fourier transform is exactly the classical Fourier transform on $\mathbb{R}^n$. Therefore, I do hope this paper will be intended to draw the attention of Analysts and Algebraist to this new way. Due to the analogues structure of two groups, it suffices to study the non commutative Fourier analysis on one of the them, for example N.

2 Plancherel Formula on N.

2.1. In the following we supply $\mathbb{R}^9$ a new structure of group by defining on $\mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 {\times} \mathbb{R} \times \mathbb{R}$ a new multiplication as:

$$\begin{aligned}
& X.Y = (x_6, x_5, x_4, x_3, x_2, t_3, t_2, x_1, t_1)(y_6, y_5, y_4, y_3, y_2, s_3, s_2, y_1, s_1) \\
= \ & ((x_6, x_5, x_4, x_3, x_2, t_3, t_2)(\rho_1(t_1)(y_6, y_5, y_4, y_3, y_2, s_3, s_2), y_1 + x_1, s_1 + t_1) \\
= \ & ((x_6, x_5, x_4, x_3)\rho_2(t_3, t_2)(y_6 + t_1y_5, y_5, y_4, y_3, s_3 + t_1s_2, s_2), x_2 + y_2, y_1 + x_1, s_1 + t_1) \\
= \ & ((x_6, x_5, x_4) + (y_6 + t_1y_5 + t_3y_4, y_5 + t_2y_4, y_4), x_3 + y_3, s_3 + t_1s_2, \\
& s_2 + t_2, x_2 + y_2, y_1 + x_1, s_1 + t_1) \\
= \ & (x_6 + y_6 + t_1y_5 + t_3y_4, x_5 + y_5 + t_2y_4, x_4 + y_4, x_3 + y_3, t_3 + s_3 + t_1s_2, \\
& y_2 + x_2, s_2 + t_2, y_1 + x_1, s_1 + t_1)
\end{aligned} \tag{9}$$

for all $(X, Y) \in \mathbb{R}^9 \times \mathbb{R}^9$. In this case the group N can be identified with the closed subgroup $\mathbb{R}^3 \times \{0\} \underset{\rho_1}{\rtimes} \mathbb{R}^2 \times\ \{0\}\ \underset{\rho_1}{\rtimes} \mathbb{R}$ of $\mathbb{R}^9$ and M with the closed subgroup $\mathbb{R}^3 \times \mathbb{R}^2 \times \{0\} {\times} \mathbb{R} \times \{0\} = \mathbb{R}^6$ of L.

2.2. Let $C^\infty(N)$ be the space of C^∞- functions on N. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{n}$ of N; which is canonically isomorphic to the algebra of all distributions on N supported by the identity element 0 of N. If $v \in \mathcal{U}$, then we can associate a right invariant differential operator (ID) on N by:

$$(ID\varphi)(X) = v * \varphi(X) = \int_N \varphi(Y^{-1}X)v(Y)dY \tag{10}$$

for any $\varphi \in C^\infty(G)$, where $dY = dydy_3dy_2dy_1$ is the Haar measure on N which is the Lebesgue measure on $\mathbb{R}^6$, $Y = (y, y_3, y_2, y_1)$, $X = (x, x_3, x_2, x_1), y = (y_6, y_5, y_4), x = (x_6, x_5, x_4)$ and $*$ denotes the convolution product on G . The mapping $v \to ID_v$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on N . For more details see [9].

Let $M = \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}$ be the vector group of N which is the direct product of $\mathbb{R}^3, \mathbb{R}^2$ and $\mathbb{R}$. we denote also by $\mathcal{U}$ the complexified universal enveloping algebra of the real Lie algebra $\underline{m}$ of M. For every $v \in \mathcal{U}$, we can associate a differential operator CD on M as follows

$$CDf(X) = \psi *_c v(X) = v *_c \psi(X) = \int_M \psi(X - Y)v(Y)dY \tag{11}$$

for any $\psi \in C^\infty(M)$, $X \in M, Y \in M$. where $*_c$ signify the convolution product on the real vector group M. The mapping $v \mapsto CD_v$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators (with constant coefficients)on M.

As in [4], we will define the Fourier transform on N. Therefore let $\mathcal{S}(N)$ be the Schwartz space of N which can be considered as the Schwartz space of $\mathcal{S}(M)$, and let $\mathcal{S}'(N)$ be the space of all tempered distributions on N.We denote by $L^1(N)$ the Banach algebra that consists of all complex valued functions on the group N, which are integrable with respect to the Haar measure of N and multiplication is defined by convolution on N, and we denote by $L^2(N)$ the Hilbert space of N.

Definition 2.1. *For every $f \in C^\infty(N)$, one can define function $\widetilde{f} \in C^\infty(\mathbb{R}^9)$ as follows:*

$$\widetilde{f}(x, x_3, x_2, t_3, t_2, x_1, t_1) = f((\rho_1(x_1)(\rho_2(x_3, x_2)(x), t_3 + x_3, t_2 + x_2)), t_1) \tag{12}$$

for all $(x, x_3, x_2, t_3, t_2, x_1, t_1) \in \mathbb{R}^9$, where $x = (x_6, x_5, x_4) \in \mathbb{R}^3$.

It results from this definition that the function $\widetilde{f}$ is invariant in the following sense:

$$\begin{aligned} & \widetilde{f}((\rho_1(h)(\rho_2(r,k)(x), x_3 - r, x_2 - k, t_3 + r, t_2 + k)), x_1 - h, t_1 + h) \\ = \; & \widetilde{f}(x, x_3, x_2, t_3, t_2, x_1, t_1) \end{aligned} \tag{13}$$

for any $(x, x_3, x_2, t_3, t_2, x_1, t_1) \in \mathbb{R}^9$, $h \in \mathbb{R}$ *and* $(r, k) \in \mathbb{R}^2$, *where* $x = (x_6, x_5, x_4) \in \mathbb{R}^3$. *So every function* $\Psi(x, x_3, x_2, x_1)$ *on* N *extends uniquely as an invariant function* $\widetilde{\Psi}(x, x_3, x_2, t_3, t_2, x_1, t_1)$ *on* $\mathbb{R}^9$.

Theorem 2.1. *For every function* $F \in C^\infty(L)$ *invariant in sense* (13) *and for every* $u \in \mathcal{U}$, *we have*

$$u \ * F(x, x_3, x_2, t_3, t_2, x_1, t_1) = u \ *_c F(x, x_3, x_2, t_3, t_2, x_1, t_1) \qquad (14)$$

for every $(x, x_3, x_2, t_3, t_2, x_1, t_1) \in L$, *where* $*$ *signifies the convolution product on* G *with respect the variables* (x, t_3, t_2, t_1) *and* $*_c$*signifies the commutative convolution product on* B *with respect the variables* (x, x_3, x_2, x_1).

Proof: In fact we have

$$\begin{aligned} & u * F(x, x_3, x_2, t_3, t_2, x_1, t_1) \\ = & \int_G F\left[(y, y_3, y_2, s)^{-1}(X, x_3, x_2, t_3, t_2, x_1, t_1)\right] u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\ = & \int_G F\left[(\rho_1(s^{-1})(y, y_3, y_2)^{-1}, -s)(x, x_3, x_2, t_3, t_2, x_1, t_1)\right] u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\ = & \int_G F[(\rho_1(s^{-1})((\rho_2(y_3, y_2)^{-1}((-y) + (x))), x_3, x_2, t_3 - y_3, t_2 - y_2), x_1, t_1 - s)] \\ & u(y, y_3, y_2, s) dy dy_3 dy_2 ds \end{aligned}$$

Since F is invariant in sense (13), then for every $(x, x_3, x_2, t_3, t_2, x_1, t_1) \in L$ we get

$$\begin{aligned} & P_u F(x, x_3, x_2, t_3, t_2, x_1, t_1) = u * F(x, x_3, x_2, t_3, t_2, x_1, t_1) \\ = & \int_G F[(\rho_1(s^{-1})(\rho_2(y_3, y_2)^{-1}(-y + x), x_3, x_2, t_3 - y_3, t_2 - y_2), x_1, t_1 - s)] \\ & u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\ = & \int_G F\left[x - y, x_3 - y_3, x_2 - y_2, t_3, t_2, x_1 - s, t\right] u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\ = & \ u *_c F(x, x_3, x_2, t_3, t_2, x_1, t_1) = Q_u F(x, x_3, x_2, t_3, t_2, x_1, t_1) \end{aligned}$$

where P_u and Q_u are the invariant differential operators on G and B respectively.

Definition 2.2. *If* $f \in \mathcal{S}(N)$, *we define the Fourier transform of* f *as follows*

$$\mathcal{F} \ f(\xi, \xi_3, \xi_2, \xi_1) = \int_N f(X, x_1) e^{- \ i \ \langle (\xi, \xi_3, \xi_2, \xi_1), (X, x_1) \rangle} dX dx_1 \qquad (15)$$

where $X = (x, x_3, x_2) \in \mathbb{R}^5, \xi = (\xi_6, \xi_5, \xi_4)$ *and* $dX = dx dx_3 dx_2$

Definition 2.3. *If $f \in \mathcal{S}(N)$, we define the Fourier transform of its invariant $\widetilde{f}$ as follows*

$$\begin{aligned} & \mathcal{F}\,\widetilde{f}(\xi,\xi_3,\xi_2,0,0,\xi_1,0) \\ = & \int_{\mathbb{R}^9} \widetilde{f}(X,t_3,t_2,x_1,t_1)e^{-\,i\,\langle(\xi,\xi_3,\xi_2,\xi_1),(X,x_1)\rangle}e^{-\,i\,\langle(\mu,\lambda,\nu),(t_3,t_2,t_1)\rangle} \\ & dXdx_1dt_3dt_2dt_1d\mu d\lambda d\nu \end{aligned} \tag{16}$$

where $\langle(\mu,\lambda,\nu),(t_3,t_2,t_1)\rangle = \mu t_3 + \lambda t_2 + \nu t_1$

Lemma 2.1 *For every $u \in \mathcal{S}(N)$, and $f \in \mathcal{S}(N)$, we have*

$$\begin{aligned} & \int_{\mathbb{R}^3} \mathcal{F}(u * \widetilde{f})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\mu d\lambda d\nu \\ = & \int_{\mathbb{R}^3} \mathcal{F}(u)(\mathcal{F}\widetilde{f})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\mu d\lambda d\nu \\ = & \mathcal{F}(\widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0)\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1) \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^3} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\mu d\lambda d\nu \\ = & \mathcal{F}(\widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0)\overline{\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1)} \end{aligned} \tag{17}$$

for any $(\xi,\xi_3,\xi_2,\xi_1,\mu,\lambda,\nu) \in \mathbb{R}^9$, where $\overset{\vee}{u}(y,y_3,y_2,s) = \overline{u(y,y_3,y_2,s)^{-1}}$.

Proof: By (17) we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \mathcal{F}(u * \widetilde{f})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\mu d\lambda d\nu \\ = & \int_{\mathbb{R}^3} \mathcal{F}(u *_c \widetilde{f})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\mu d\lambda d\nu \\ = & \int_{\mathbb{R}^3} \mathcal{F}(u)(\mathcal{F}\widetilde{f})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\mu d\lambda d\nu \\ = & \mathcal{F}(\widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0)\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1) \end{aligned}$$

So

$$\begin{aligned} & \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0) = \mathcal{F}(\overset{\vee}{u} *_c \widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0) \\ = & \mathcal{F}(\widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0)\overline{\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1)} \end{aligned}$$

Theorem 2.1 (*Plancherel formula*). *For any $f \in L^1(N) \cap L^2(N)$, we* *get*

$$f \ast \overset{\widetilde{\vee}}{f}(0,0,0,0,0,0,0,0,0) = \int\limits_{N} |f(x,x_3,x_2,x_1)|^2 \, dx dx_3 dx_2 dx_1$$

$$= \int\limits_{\mathbb{R}^6} |\mathcal{F}f(\xi,\xi_3,\xi_2,\xi_1)|^2 \, d\xi d\xi_3 d\xi_2 d\xi_1 \qquad (18)$$

where

$$\overset{\widetilde{\vee}}{f}(x,x_3,x_2,t_3,t_2,x_1,t_1) = \overset{\vee}{f}((\rho_1(x_1)(\rho_2(x_3,x_2)(x)),t_3,t_2),x_1+t_1)$$
$$= \overline{f((\rho_1(x_1)(\rho_2(x_3,x_2)(x)),t_3,t_2),x_1+t_1)^{-1})} \qquad (19)$$

Proof: If $f \in \mathcal{S}(N)$. Then we have

$$f \ast \overset{\widetilde{\vee}}{f}(0,0,0,0,0,0,0,0,0))$$
$$= \int\limits_{N} \overset{\widetilde{\vee}}{f}\left[(x,x_3,x_2,x_1)^{-1}(0,0,0,0,0,0,0,0,0)\right] f(x,x_3,x_2,x_1) dx dx_3 dx_2 dx_1$$
$$= \int\limits_{N} \overset{\widetilde{\vee}}{f}\left[\rho_1(x_1^{-1})((x,x_3,x_2)^{-1}(0,0,0,0,0,0,0,0)),0-x_1\right] f(x,x_3,x_2,x_1) dx dx_3 dx_2 dx_1$$
$$= \int\limits_{N} \overset{\widetilde{\vee}}{f}\left[(\rho(x_1^{-1})((\rho_2(x_3,x_2)^{-1}((-x)+(0,0,0))),0,0,0-x_3,0-x_2),0,-x_1\right]$$
$$f(x,x_3,x_2,x_1) dx dx_3 dx_2 dx_1$$
$$= \int\limits_{N} \overset{\widehat{\vee}}{f}\left[(\rho(x_1^{-1})(\rho_2(x_3,x_2)^{-1}(-x)),0,0,-x_3,-x_2),0,-x_1\right] f(x,x_3,x_2,x_1) dx dx_3 dx_2 dx_1$$
$$= \int\limits_{N} \overset{\widetilde{\vee}}{f}\left[(\rho(x_1^{-1})(\rho_2(x_3,x_2)^{-1}(-x)),-x_3-x_2),-x_1\right] f(x_6,x_5,x_4,x_3,x_2,x_1) dx dx_3 dx_2 dx_1$$
$$= \int\limits_{N} \overset{\vee}{f}\left[(x,x_3 x_2,x_1)^{-1}\right] f(x,x_3,x_2,x_1) dx dx_3 dx_2 dx_1$$
$$= \int\limits_{N} \overline{f(x,x_3,x_2,x_1)} f(x,x_3,x_2,x_1) dx dx_3 dx_2 dx_1 = \int\limits_{\mathbb{R}^6} |f(x,x_3,x_2,x_1)|^2 \, dx dx_3 dx_2 dx_1$$

Now equation (17), give us

$$
\begin{aligned}
& f \ * \widetilde{\check{f}}(0,0,0,0,0,0,0,0,0) \\
= & \int\limits_{\mathbb{R}^9} \mathcal{F}(f * \widetilde{\check{f}})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\xi d\xi_3 d\xi_2 d\xi_1 d\mu d\lambda d\nu \\
= & \int\limits_{\mathbb{R}^9} \mathcal{F}(f \star_c \widetilde{\check{f}})(\xi,\xi_3,\xi_2,\mu,\lambda,\xi_1,\nu)d\xi d\xi_3 d\xi_2 d\xi_1 d\mu d\lambda d\nu \\
= & \int\limits_{\mathbb{R}^6} \mathcal{F}[\widetilde{\check{f}}](\xi,\xi_3,\xi_2,0,0,\xi_1,0)\mathcal{F}(f) \ (\xi,\xi_3,\xi_2,\xi_1)d\xi d\xi_3 d\xi_2 d\xi_1 \\
= & \int\limits_{\mathbb{R}^{18}} \widetilde{\check{f}}(y,y_3,y_2,0,0,y_1,0)e^{- \ i \ \langle \xi, Y\rangle} f \ (x,x_3,x_2,x_1)e^{- \ i \ \langle \xi, X\rangle} \\
& dy dy_3 dy_2 dy_1 dx dx_3 dx_2 dx_1 d\xi d\xi_3 d\xi_2 d\xi_1 \\
= & \int\limits_{\mathbb{R}^{18}} \check{f}((\rho(y_1)(\rho_2(y_3,y_2)(y)),y_3,y_2),y_1)e^{- \ i \ \langle \xi, Y\rangle} f \ (x,x_3,x_2,x_1)e^{- \ i \ \langle \xi, X\rangle} \\
& dy dy_3 dy_2 dy_1 dx dx_3 dx_2 dx_1 d\xi d\xi_3 d\xi_2 d\xi_1 \\
= & \int\limits_{\mathbb{R}^{18}} \overline{f(-y,-y_3,-y_2,-y_1)}e^{- \ i \ \langle (\xi,\xi_3,\xi_2,\xi_1),(y,y_3,y_2,y_1)\rangle} f \ (x,x_3,x_2,x_1)e^{- \ i \ \langle (\xi,\xi_3,\xi_2,\xi_1),(x,x_3,x_2,x_1)\rangle} \\
& dy dy_3 dy_2 dy_1 dx dx_3 dx_2 dx_1 d\xi d\xi_3 d\xi_2 d\xi_1 \\
= & \int\limits_{\mathbb{R}^6} \overline{\mathcal{F}(f)} \ (\xi,\xi_3,\xi_2,\xi_1)\mathcal{F}(f) \ (\xi,\xi_3,\xi_2,\xi_1)d\xi d\xi_3 d\xi_2 d\xi_1 \\
= & \int\limits_{\mathbb{R}^6} \left|\mathcal{F}f(\xi,\xi_3,\xi_2,\xi_1)\right|^2 d\xi d\xi_3 d\xi_2 d\xi_1 = \int\limits_{N} \left|f(x,x_3,x_2,x_1)\right|^2 dx dx_3 dx_2 dx_1
\end{aligned}
$$

which is the Plancheral's formula on N.

Corollary 2. 1. *In equation* (18), *replace the second* f by g *we obtain the Parseval formula on* N

$$
\begin{aligned}
& \int\limits_{N} \overline{f(x,x_3,x_2,x_1)}g(x,x_3,x_2,x_1)dx dx_3 dx_2 dx_1 \\
= & \int\limits_{\mathbb{R}^6} \overline{\mathcal{F}f(\xi,\xi_3,\xi_2,\xi_1)}\mathcal{F}g(\xi,\xi_3,\xi_2,\xi_1)d\xi d\xi_3 d\xi_2 d\xi_1
\end{aligned}
$$

3 Division of Distributions on N.

If we consider the group N as a subgroup of L, then $\widetilde{f} \in \mathcal{S}(N)$ for x_1, x_2 and x_3 are fixed, and if we consider M as a subgroup of $\mathbb{R}^9$, then $\widetilde{f} \in \mathcal{S}(M)$ for

t_1, t_2 and t_3 fixed. This being so; denote by $\mathcal{S}_E(\mathbb{R}^9)$ the space of all functions $\Phi(x, x_3, x_2, t_3, t_2, x_1, t_1) \in C^\infty(\mathbb{R}^9)$ such that $\Phi(x, x_3, x_2, t_3, t_2, x_1, t_1) \in \mathcal{S}(N)$ for x_1, x_2 and x_3 are fixed, and $\Phi(x, x_3, x_2, t_3, t_2, x_1, t_1) \in \mathcal{S}(M)$ for $t_1, t_2$and t_3 fixed. We equip $\mathcal{S}_E(\mathbb{R}^9)$ with the natural topology defined by the seminormas:

$$\Phi \to \sup_{(x,x_3,x_2,x_1)\in M} |Q(x, x_3, x_2, t_3, t_2, x_1, t_1)P(D)\Phi(x, x_3, x_2, t_3, t_2, x_1, t_1)| \qquad t_3, t_2, t_1 \ fixed$$

$$\Phi \to \sup_{(x,t_3,t_2,t_1)\in N} |R(x, x_3, x_2, t_3, t_2, x_1, t_1)S(D)\Phi(x, x_3, x_2, t_3, t_2, x_1, t_1)| \qquad x_3, x_2, x_1 \ fixed \tag{20}$$

where P, Q, R and S run over the family of all complex polynomials in 9 variables. Let $\mathcal{S}_E^I(L)$ be the subspace of all functions $\psi \in \mathcal{S}_E(\mathbb{R}^9)$, which are invariant in sense (13), then we have the following result.

Lemma 3. 1. *Let* $u \in \mathcal{U}$ *and* CD_u *be the invariant differential operator on the group* M, which is associated to u, *acts on the variables* $(x, x_3, x_2, x_1) \in M$, then we have

(i) The mapping $f \mapsto \widetilde{f}$ *is a topological isomorphism of* $\mathcal{S}(N)$ *onto* $\mathcal{S}_E^I(\mathbb{R}^9)$.

(ii) The mapping $\Phi \mapsto CD_u\Phi$ *is a topological isomorphism of* $\mathcal{S}_E^I(\mathbb{R}^9)$ *onto its image*

Proof: (i) In fact $\sim$ is continuous and the restriction mapping $\Phi \mapsto R\Phi$ on N is continuous from $\mathcal{S}_E^I(\mathbb{R}^9)$ into $\mathcal{S}(N)$ that satisfies $R\circ \sim = Id_{\mathcal{S}(N)}$ and $\sim \circ R = Id_{\mathcal{S}_E^I(\mathbb{R}^9)}$, where $Id_{\mathcal{S}(N)}$ $(resp.\ Id_{\mathcal{S}_E^I(\mathbb{R}^9)})$ is the identity mapping of $\mathcal{S}(N)$ $(resp.\ \mathcal{S}_E^I(\mathbb{R}^9))$ and N is considered as a subgroup of $\mathbb{R}^9$.

To prove(ii) we refer to [12, $P.313-315$] and his famous result that is: "*Any invariant differential operator on* M, *is a topological isomorphism of* $S(M)$ *onto its image*" From this result, we obtain that

$$CD_u : \mathcal{S}_E(\mathbb{R}^9) \to \mathcal{S}_E(\mathbb{R}^9) \tag{21}$$

is a topological isomorphism and its restriction on $\mathcal{S}_E^I(\mathbb{R}^9)$ is a topological isomorphism of $\mathcal{S}_E^I(\mathbb{R}^9)$ onto its image. Hence the theorem is proved.

In the following we will prove that every invariant differential operator on N has a tempered fundamental solution. As in the introduction, we will consider the two invariant differential operators ID_u and CD_u, the first on the group $N = \mathbb{R}^3 \times \{0\} \times \mathbb{R}^2 \times \{0\} \times \mathbb{R}$, and the second on the abelian vector group $M = \mathbb{R}^3 \times \mathbb{R}^2 \times \{0\} \times \mathbb{R} \times \{0\}$. Our main result is:

Theorem 3.1. *Every nonzero invariant differential operator on* N *has a tempered fundamental solution*

Proof: For every function $\psi \in C^\infty(\mathbb{R}^9)$ invariant in sensen (13) and for every $u \in \mathcal{U}$, we have

$$u * \psi(x, x_3, x_2, t_3, t_2, x_1, t_1) = u *_c \psi(x, x_3, x_2, t_3, t_2, x_1, t_1) \tag{22}$$

for every $(x, x_3, x_2, t_3, t_2, x_1, t_1) \in L$, where $*$ signifies the convolution product on N with respect the variables (x, t_3, t_2, t_1) and $*_c$signifies the commutative convolution product on M with respect the variables (x, x_3, x_2, x_1). In fact we have

$$\begin{aligned}
& ID_u * \psi(x, x_3, x_2, t_3, t_2, x_1, t_1) \\
= & \int\limits_N \psi\left[(y, y_3, y_2, s)^{-1}(X, x_3, x_2, t_3, t_2, x_1, t_1)\right] u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\
= & \int\limits_N \psi\left[(\rho_1(s^{-1})(y, y_3, y_2)^{-1}, -s)(x, x_3, x_2, t_3, t_2, x_1, t_1)\right] u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\
= & \int\limits_N \psi[(\rho_1(s^{-1})((\rho_2(y_3, y_2)^{-1}((-y) + (x))), x_3, x_2, t_3 - y_3, t_2 - y_2), x_1), t_1 - s] \\
& u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\
= & \int\limits_N \psi[((\rho_1(s^{-1})(\rho_2(y_3, y_2)^{-1}(-y + x), x_3, x_2, t_3 - y_3, t_2 - y_2), x_1), t_1 - s)] \\
& u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\
= & \int\limits_M \psi\left[x - y, x_3 - y_3, x_2 - y_2, t_3, t_2, x_1 - s, t\right] u(y, y_3, y_2, s) dy dy_3 dy_2 ds \\
= & u *_c \psi(x, x_3, x_2, t_3, t_2, x_1, t_1) = CD_u \psi(x, x_3, x_2, t_3, t_2, x_1, t_1) \qquad (23)
\end{aligned}$$

for all $(x, x_3, x_2, t_3, t_2, x_1, t_1) \in \mathbb{R}^9$. By Lemma 2.1, the mapping $\psi \mapsto CD_u\psi$ is a topological isomorphism of $\mathcal{S}_E^I(\mathbb{R}^9)$ onto its image, then the mapping $\psi \mapsto ID_u\psi$ is a topological isomorphism of $\mathcal{S}_E^I(\mathbb{R}^9)$ onto its image. Since

$$R(ID_u\psi)(x, x_3, x_2, t_3, t_2, x_1, t_1) = ID_u(R\psi)(x, x_3, x_2, t_3, t_2, x_1, t_1) \qquad (24)$$

so the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{S}_E^I(\mathbb{R}^9) & \xrightarrow{ID_u} & P_u\mathcal{S}_E^I(\mathbb{R}^9) \\
\sim\uparrow\downarrow R & & \downarrow R \\
\mathcal{S}(N) & \xrightarrow{ID_u} & P_u\mathcal{S}(N)
\end{array}$$

Hence the mapping $\psi \mapsto ID_u\psi$ is a topological isomorphism of $\mathcal{S}(N)$ onto its image. So the transpose ${}^tID_u$ of ID_u is a continuous mapping of $\mathcal{S}'(N)$ onto $\mathcal{S}'(N)$. This means that for every tempered distribution T on N there is a tempered distribution E on N such that

$$ID_uF = T \qquad (25)$$

Indeed the Dirac measure δ belongs to $\mathcal{S}'(N)$.fundamental solution on N for any element $u \in \mathcal{U}$.

As in [6] we show how Atiyah method [1], can be generalized for our group N to obtain a tempered solution by the following theorem.

Theorem 3.2. *Every invariant differential operator on N which is not identically 0 has a tempered fundamental solution.*

Proof: For each complex number s with positive real part, we can define a distribution T^s on L by:

$$\langle T^s, f\rangle = \int_{\mathbb{R}^6} \left[|\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1)|^2\right]^s \mathcal{F}\ f(\xi,\xi_3,\xi_2,\xi_1)d\xi d\xi_3 d\xi_2 d\xi_1 \tag{26}$$

for each $f \in \mathcal{S}(\mathbb{R}^9)$, where $\xi = (\xi_6,\xi_5,\xi_4)$ and $d\xi = d\xi_6 d\xi_5 d\xi_4$. By Atiyah theorem [1], the function $s \mapsto T^s$ has a meromorphic continuation in the whole complex plan, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group N . Now we can define another distribution $\widetilde{T^s}$ as follows.

$$\begin{aligned}\left\langle \widetilde{T^s}, f\right\rangle &= \left\langle T^s, \widetilde{f}\right\rangle \\ &= \int_{\mathbb{R}^9} \left[|\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1)|^2\right]^s \mathcal{F}(\widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0)d\xi d\xi_3 d\xi_2 d\xi_1\end{aligned} \tag{27}$$

for any $f \in \mathcal{S}(\mathbb{R}^9)$ and s is a complex number, with real(s) is positive. Note that the distribution $\widetilde{T^s}$ is invariant in sense (13), so we have

$$\begin{aligned}&\left\langle \widetilde{u * \overset{\vee}{u} *_c T^s}, f\right\rangle = \left\langle u * \overset{\vee}{u} *_c T^s, \widetilde{f})\right\rangle = \left\langle T^s, u *_c \overset{\vee}{u}_c * \widetilde{f})\right\rangle \\ &= \int_L \left[|\mathcal{F}(u)(\xi,\xi_3,\xi_2,\xi_1)|^2\right]^{s+1} \mathcal{F}(\widetilde{f}))(\xi,\xi_3,\xi_2,0,0,\xi_1,0)d\xi d\xi_3 d\xi_2 d\xi_1\end{aligned}$$

where $\overset{\vee}{u}(y,y_3,y_2,y_1) = \overline{u(y,y_3,y_2,s)^{-1}}$ and

$$\begin{aligned}&u *_c f(x,x_3,x_2,x_1) \\ &= \int_G f(x-y,x_3-y_3,x_2-y_2,x_1-y_1)u(-y,-y_3,-y_2,-y_1)dydy_3dy_2dy_1\end{aligned}$$

is the commutative convolution product on M. By Lemma 2.1, we get:

$$\begin{aligned}&\left\langle \widetilde{u * \overset{\vee}{u} *_c T^s}, f\right\rangle \\ &= \int_{\mathbb{R}^9} \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi,\xi_3,\xi_2,\xi_1)\right|^2\right]^{s+1} \mathcal{F}(\widetilde{f})(\xi,\xi_3,\xi_2,0,0,\xi_1,0)d\xi d\xi_3 d\xi_2 d\xi_1\end{aligned}$$

Hence

$$\widetilde{u * \overset{\vee}{u} *_c T^s} = \widetilde{T^s} \tag{28}$$

In view of the invariance (13), the restriction of the distributions $\widetilde{u * \overset{\vee}{u} *_c T^s} = \widetilde{T^{s+1}}$ on the sub-group $\mathbb{R}^3 \times \{0\} \underset{\rho_2}{\rtimes} \mathbb{R}^2 \times \{0\} \underset{\rho_1}{\rtimes} \mathbb{R} \simeq N$ are nothing but the distributions

$$u * \overset{\vee}{u} *_c T^s = T^{s+1}. \tag{29}$$

The distribution T^s can be expanded a round $s = -1$ in the form

$$T^s = \sum_{j=-6}^{\infty} \beta_j (s+1)^j \tag{30}$$

where each β_j is a distribution on N . But $u * \overset{\vee}{u} *_c T^s = T^{s+1}$ can not have a pole at $s = -1$ (since $T^0 = \delta_N$) and so we must have:

$$\begin{aligned} u * \overset{\vee}{u} *_c \beta_j &= 0 \quad \text{for} \quad j < 0 \\ u * \overset{\vee}{u} *_c \beta_0 &= \delta_N \end{aligned} \tag{31}$$

Whence the theorem.

4 Ideals Algebra $L^1(N)$.

We refer here to [5] for the characterization all left ideals in the Banach algebra $L^1(N)$.

Lemma 4.1. (i) *The mapping* Θ *from* $\widetilde{L^1(N)}|_M$ *to* $\widetilde{L^1(N)}|_N$ *defined by*

$$\begin{aligned} \widetilde{\phi}|_M \ (x, x_3, x_2, 0, 0, x_1, 0) &\to \Theta(\widetilde{\phi}|_M)(x, 0, 0, x_3, x_2, 0, x_1) \\ &= \widetilde{\phi}|_N(x, 0, 0, x_3, x_2, 0, x_1) \end{aligned} \tag{32}$$

is a topological isomorphism
(ii) *For every* $w \in L^1(N)$ *and* $\phi \in L^1(N)$, *we obtain*

$$\begin{aligned} \Gamma(w \ *_c \widetilde{\phi}|_M)(x, 0, 0, x_3, x_2, 0, x_1) &= w \ * \widetilde{\phi}|_N(x, 0, 0, x_3, x_2, 0, x_1) \\ &= w * \phi(x, x_3, x_2, x_1) \end{aligned} \tag{33}$$

where $x = (x_6, x_5, x_4)$, $y = (y_6, y_5, y_4)$, and

$$\begin{aligned} & (w \ *_c \widetilde{\phi}|_M)(x, x_3, x_2, 0, 0, x_1, 0) \\ = & \int_M \widetilde{\phi}\,[x - y, x_3 - y_3, x_2 - y_2, 0, 0, x_1 - y_1, 0]\, w(y, y_3, y_2, y_1) \\ & dy dy_3 dy_2 dy_1 \ , \qquad \phi \in L^1(N) \end{aligned} \tag{34}$$

Proof: (i) The mapping Θ is continuous and has an inverse Θ^{-1} given by

$$\begin{aligned} \widetilde{\phi}|_N \ (x, 0, 0, x_3, x_2, 0, x_1) &\to \Theta^{-1}(\widetilde{\phi}|_N \ )(x, x_3, x_2, 0, 0, x_1, 0) \\ &= \widetilde{\phi}|_N \ (x, x_3, x_2, 0, 0, x_1, 0) \end{aligned} \tag{35}$$

(ii) It is enough to see for every $\phi \in L^1(N)$

$$\begin{aligned}
& \Theta(w *_c \widetilde{\phi}|_M)(x,0,0,x_3,x_2,0,x_1) \\
= & \int_M \widetilde{\phi}|_M\,[x-y,-y_3,-y_2,x_3,x_2,-y_1,x_1]\,w(y,y_3,y_2,0,y_1) \\
& dydy_3dy_2dy_1 \\
& \int_M \widetilde{\phi}|_M\,[x-y,-y_2,x_3,x_2,-y_1,x_1]\,w(y,y_3,y_2,0,y_1) \\
& dydy_3dy_2dy_1 \\
= & \int_N \phi[\rho_1(x^{-1})(\rho_2(y_3,y_2)^{-1}(x-y),x_3-y_3,x_2-y_2),x_1-y_1] \\
& w(x,y_3,y_2,s)dydy_3dy_2dy_1 \\
= & w*\phi(x,x_3,x_2,x_1), \qquad \phi \in L^1(N) \qquad (36)
\end{aligned}$$

If I is a subalgebra of $L^1(N)$, we denote by $\widetilde{I}$ its image by the mapping $\sim$. Let $J = \widetilde{I}\,|_M$. Our main result is:

Theorem 4.1. *Let I be a subalgebra of $L^1(N)$, then the following conditions are equivalents.*

(i) $J = \widetilde{I}\,|_M$ is an ideal in the Banach algebra $L^1(M)$.

(ii) I is a left ideal in the Banach algebra $L^1(N)$.

Proof: (i) implies (ii) Let I be a subspace of the space $L^1(M)$ such that $J = \widetilde{I}|_M$ is an ideal in $L^1(M)$, then we have:

$$w *_c \widetilde{I}\,|_M(x,x_3,x_2,0,0,x_1,0) \subseteq \widetilde{I}\,|_M(x,x_3,x_2,0,0,x_1,0) \qquad (37)$$

for any $w \in L^1(M)$ and $(x,x_3,x_2,x_1) \in M$, where

$$\begin{aligned}
& w *_c \widetilde{I}\,|_M(x,x_3,x_2,0,0,x_1,0) \\
= & \left\{ \begin{array}{c} \int_M \widetilde{\phi}\,[x-y,x_2-y_2,0,0,x_1-y_1,0]\,w(y,y_3,y_2,y_1) \\ dydy_3dy_2dy_1, \qquad \phi \in L^1(N) \end{array} \right\}
\end{aligned}$$

It shows that

$$w *_c \widetilde{\phi}\,|_M(x,x_3,x_2,0,0,x_1,0) \in \widetilde{I}\,|_M(x,x_3,x_2,0,0,x_1,0) \qquad (38)$$

for any $\widetilde{\phi} \in \widetilde{I}$. Then we get

$$\begin{aligned}
& \Theta(w *_c \widetilde{\phi}|_M)(x,0,0,x_3,x_2,0,x_1) \\
= & u\ * \widetilde{F}(x,0,0,x_3,x_2,0,x_1) \in \Theta(\widetilde{I}\,|_M)(x,0,0,x_3,x_2,0,x_1) \\
= & \widetilde{I}\,|_N(x,0,0,x_3,x_2,0,x_1) = I(x,x_3,x_2,x_1) \qquad (39)
\end{aligned}$$

It is clear that (ii) implies (i).

Corollary 4.1. *Let I be a subalgebra of the space $L^1(N)$ and $\widetilde{I}$ its image by the mapping $\sim$ such that $J = \widetilde{I}|_M$ is an ideal in $L^1(N)$, then the following conditions are verified.*

(i) J is a closed ideal in the algebra $L^1(M)$ if and only if I is a left closed ideal in the algebra $L^1(N)$.

(ii) J is a maximal ideal in the algebra $L^1(M)$ if and only if I is a left maximal ideal in the algebra $L^1(N)$.

(iii) J is a prime ideal in the algebra $L^1(M)$ if and only if I is a left prime ideal in the algebra $L^1(N)$.

(iv) J is a dense ideal in the algebra $L^1(M)$ if and only if I is a left dense ideal in the algebra $L^1(N)$.

The proof of this corollary results immediately from theorem **4.1**

5 Conclusion.

5.1. The Fourier transform has a natural generalization to our groups and many of the classical results can be extended. In fact our results obtained in this paper show the beauty of the natural extension of the Fourier transform to the non commutative and non compact Lie groups. Its powerful can be seen also through the following astonishing results.

Let H be the 3−dimensional Heisenberg group consisting of all matrices of the form

$$H = \begin{pmatrix} 1 & x_1 & 0 & x_6 \\ 0 & 1 & 0 & x_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

It is easy to show that the group H is a normal subgroup of the group N. The most interesting result that can be deduced from my work [**4**, **8**] for this group is: "*Any invariant differential operator on H is globally solvable*".

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