An Effect of Total Domination upon Edge-Vertex Removal of Graphs

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***Abstract* - Graphs which are critical with respect to the total domination have been studied by several authors. This paper is concerned about what happens to total domination number when an edge or vertex is removed from the graph. Several references are used to include all the results of this paper.**

Keywords: Total Dominating set, Minimum Total dominating set, Total domination number, Subgraph, Pendant Edge, Star, Double Star.

INTRODUCTION

Total Domination is undefined for graphs with isolated vertices. The graph G is a total domination edge critical or γ_t - critical for short, if the removal of any edge in the graph changes the total domination number, that is $\gamma_t(G-e) \neq \gamma_t(G)$ for every edge $e \in E(G)$. Note that removing an edge from a graph cannot decrease the total domination number. Hence if G is γ_t - critical, then $\gamma_t(G-e) > \gamma_t(G)$ for every edge $e \in E(G)$. Thus, every edge in a γ_t - critical graph is a critical edge.

Let G be a graph. A vertex v of G is said to be a pendant vertex (or leaf) if and only if it has degree 1. An edge of a graph is said to be pendant if one of its vertices is a pendant vertex.

A tree $T \in \mathcal{T}$ if T is a nontrivial star, or a double star, or if T can be obtained from a subdivided star $K^*_{1,k}$, where $K \geq 2$, by adding zero or more pendant edges to the non-leaf vertices of $K^*_{1,k}$. A tree containing exactly one vertex that is not a pendant vertex is called a star. The double star $DS(m,n)$ is a tree of diameter three such that there are m pendant edges on one end of path P_2 and n pendant edges on the other end.

1. EDGE CRITICAL GRAPHS

This Section computes a characterization of γ_t-critical graph.

Proposition 1.1 If S is a minimal total dominating set of connected graph G, then for each vertex $v \in S$, $|epn(v,S)| \geq 1$ or $G[S\backslash\{v\}]$ contains an isolated vertex.

Lemma 1.1 If G is a γ_t-critical graph, then for every $\gamma_t(G)$-set S, $G[S]$ is a galaxy of nontrivial stars.

Proof Let S be any $\gamma_t(G)$ -set in the γ_t-critical graph G, and let G_S=$G[S]$. Let e be an arbitrary edge in G_S. If both ends of e have degree at least 2 in G_S, then S is a total dominating set in $G-e$, and so $\gamma_t(G-e) \leq |S| = \gamma_t(G)$, contradicting the fact that G is γ_t-critical. Hence at least one end of the edge e is a leaf in G_S, implying that G_S is a galaxy of nontrivial stars.

Theorem 1.2 A connected graph G is γ_t-critical if and only if $G \in \mathcal{T}$.

Proof Assume that $G=(V,E)$ is γ_t-critical. Let S be any $\gamma_t(G)$-set. By Lemma 1.1, $G[S]$ is a galaxy of nontrivial stars. If v is a leaf in $G[S]$ and v is adjacent to a vertex of degree at least two in $G[S]$, then by Proposition 1.1,$|epn(v,S)| > 1$. Thus, v has an external private neighbor and is therefore adjacent to at least one vertex in $V\backslash S$. For every edge e of G, if S is a total dominating set in $G-e$,then $\gamma_t(G-e) \leq |S| = \gamma_t(G)$,contradicting the fact that G is γ_t-critical. Hence for every edge e of G, the set S is not a total dominating set in $G-e$. This implies that $V\backslash S$ is an independent set and that each vertex in $V\backslash S$ is adjacent to exactly one vertex of S and is therefore a leaf in G. Thus since G is connected, the subgraph $G[S]$ is connected. Hence, $G[S]$ is a star. If $G[S]=K_2$, then G is either a star or a double star, and so $G \in \mathcal{T}$. Hence assume that $G[S]$ is a star $K_{1,k}$ where $k \geq 2$. As observed earlier, each leaf in the star $G[S]$ is adjacent to at least one vertex in $V\backslash S$. Let L denote a set of k vertices in $V\backslash S$ that dominate the set of k leaves in $G[S]$. Then, $G[S \cup L] = K^*_{1,k}$ and G can be obtained from this subdivided star by adding zero or more pendant edges to each vertex of S. Thus, $G \in \mathcal{T}$.

Now, assume that $G \in \mathcal{T}$. Let $G=(V,E)$and let $e \in E$. If e is incident with a leaf in G, then $\gamma_t(G-e) = \infty$, and so e is a critical edge. Hence, assume that e is not incident with a leaf in G. In particular, G is not a star. If G is a double star, then e joins the two central vertices of G. Thus, $\gamma_t(G-e)=4$ while $\gamma_t(G)=2$, and so e is a critical edge. Hence assume that G is not a double star. Thus, G is obtained from a star $T=K^*_{1,k}$, for some $k \geq 2$, by adding at least one pendant edge to each leaf of T and adding zero or more pendant edges to the center v of the star T. Every edge in the set $E\backslash E(T)$ is incident with a leaf in G. Hence, by earlier assumptions,$e \in E(T)$. But then $\gamma_t(G-e) = k+2$ while $\gamma_t(G)=k+1$ (irrespective of whether v is a support vertex of G). Hence, once again, the edge e is a critical edge. Therefore, G is γ_t-critical.

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2. VERTEX CRITICAL GRAPHS

When a vertex is removed from a graph, the total domination number may increase, decrease or remains unchanged. Here three sets namely V_T^+, V_T^- and V_T^0 in a graph are defined. All graphs are simple and a total dominating set contains at least two vertices. Also graphs considered have no isolated vertices.

If G is a graph then $V(G)$ will denote the vertex set of G. If v is a vertex of G, then $G - v$ will denote the subgraph obtained by removing the vertex v from the graph. Let G be a graph. A subset S of $V(G)$ is said to be a total dominating set in the graph G if any vertex v of the graph G is adjacent to at least one vertex of the set S. A total dominating set S in the graph G is said to be a minimal total dominating set in the graph G if for any vertex v of S, $S - v$ is not a total dominating set in the graph G.

Definition: Let G be any graph. Then,

$V_T^i = \{v \in V(G) \,/\, G - v$ has an isolated vertex$\}$.

$V_T^+ = \{v \in V(G) \,/\, \gamma_T(G - v) > \gamma_T(G)\}$.

$V_T^- = \{v \in V(G)/\, \gamma_T(G - v) < \gamma_T(G)\}$.

$V_T^0 = \{v \in V(G) \,/\gamma_T(G - v) = \gamma_T(G)\}$.

Theorem 2.1 Let G be any graph and $v \in V(G)$ such that $v \notin V_T^i$. Then $v \in V_T^+$ if and only if the following conditions are satisfied:

(a) Every γ_T-set of the graph G contains v.

(b) If S is a subset of $V(G) - N[v]$ such that $|S| = \gamma_T(G)$, then S is not a total dominating set in $G - v$.

Proof The proof is standard and hence it is omitted.

Theorem 2.2 Let G be any graph and let $v \in V(G)$ with $v \notin V_T^i$. If for any $w \in N(v)$, the subgraph induced by $N(w)$ is complete, then $v \notin V_T^-$.

Proof Let G be any graph and $v \in V(G)$ such that $v \notin V_T^i$. Let the subgraph induced by $N(w)$ be complete for every $w \in N(v)$. To prove $v \notin V_T^-$. Suppose $v \in V_T^-$. Therefore, there is a minimum total dominating set S in the graph G not containing v and a vertex z in S such that $TPr[z, S] = \{v\}$. Therefore, $v \notin S$ is adjacent to only one vertex z in S. Therefore, $z \in N(v)$. Also, $z \in S$ and S is a total dominating set in the graph G. Then, z is adjacent to some vertex x in S. Therefore both x and v are in $N(z)$. But, the subgraph induced by $N(z)$ is complete. Therefore, v is adjacent to a vertex x. Therefore, v is adjacent to two distinct vertices x, z of S. Therefore, $v \notin TPr[z, S]$, a contradiction to the fact that $TPr[z, S] = \{v\}$. Therefore, the assumption is wrong. Therefore, $v \notin V_T^-$.

Theorem 2.3 Suppose $v \in V_T^+$ and S is a minimum total dominating set in the graph G containing v with $v \notin V_T^i$. Then the following statements are true:

(i) If $TPr[v, S] = \{w\}$, then $w \notin S$.

(ii) $TPr[v, S]$ contains at least two vertices different from v.

(iii) If $TPr[v, S]$ contains more than one vertex and w_1, w_2, are such adjacent vertices, then at least one $w_i \notin S$.

Proof (i) Let G be any graph and $v \in V(G)$ such that $v \notin V_T^i$. Let $v \in V_T^+$ and S be a minimum total dominating set in the graph G containing v. Suppose $TPr\,[v, S] = \{w\}$. To show that $w \notin S$. Suppose $w \in S$. Therefore, w is adjacent to only v in S. But $v \in V_T^+$ imply $v \notin V_T^i$. Therefore, the graph $G - v$ does not contain any isolated vertex. Therefore, find some vertex $z \notin S$ such that w is adjacent to z(because if $z \in S$,then $w \notin TPr\,[v, S]$). Let $S_1 = S - \{v\} \cup \{z\}$.

Case 1. If $x = w$, then x is adjacent to a vertex $z \in S_1$.

Case 2. If $x \neq w \in V(G - v)$, then x is adjacent to some vertex $y \in S$ different from v (because S is a total dominating set in the graph G and therefore if x is adjacent to only $v \in S$, then $TPr[v, S]$ contains x different from w, a contradiction to the fact that $TPr\,[v, S] = \{w\}$). That is if $x \neq w \in V(G - v)$, then x is adjacent to some vertex $y \in S_1$.

Case 3. If $x = v$, then x is adjacent to a vertex $w \in S_1$. Thus, from all cases can say that S_1 is a total dominating set in the graph G not containing v with $|S_1| = \gamma_T(G)$. That is, S_1 is a γ_T-set in the graph G not containing v, a contradiction to the fact that $v \in V_T^+$. Therefore, $w \notin S$.

(ii) Suppose $v \in V_T^+$ and S is a minimum total dominating set in G containing v. Therefore, Let G be a graph and S be a subset of $V(G)$. A total dominating set S in the graph G is a minimal total dominating set in G if and only if for every vertex $v \in S$, $TPr[v, S] \neq \emptyset$. Suppose, $TPr[v, S] = \{w\}$. Therefore, $w \notin S$(by Theorem 2.3(i)). Also $v \notin V_T^i$. Therefore, the graph $G - v$ does not contain any isolated vertex. Therefore, there is a vertex $z \neq v$ in $v(G)$ such that w is adjacent to z.(because if w is adjacent to only v in G, then $G - v$ contains an isolated vertex w and therefore $v \in V_T^i$, a contradiction to the fact that $v \notin V_T^i$). Also $z \notin S$ (because if $z \in S$, then w is adjacent to two distinct vertices z, v of S and therefore $w \notin TPr\,[v, S]$, a contradiction to the fact that $TPr\,[v, S] = \{w\}$). But S is a total dominating set in G. Therefore, z is adjacent to some vertex $x \in S$ different from v (because if z is adjacent to only v in S, then $TPr\,[v, S]$ contains one element z different from w, a contradiction to the fact that $TPr\,[v, S] = \{w\}$). Let $S_1 = S - \{v\} \cup \{z\}$.

Case 1. If $y = w$, then y is adjacent to a vertex $z \in S_1$.

Case 2. If $y \neq w \in V(G - v)$, then y is adjacent to some vertex $p \in S$ different from v (because S is a total dominating set in the graph G and if y is adjacent to only $v \in S$, then $TPr\,[v, S]$ contains y different from w, a contradiction to the fact that $TPr\,[v, S] = \{w\}$). That is, if $y \neq w \in V(G - v)$, then y is adjacent to some vertex $p \in S_1$.

Case 3. If $y = v$, then y is adjacent to some vertex p in S (because S is a total dominating set in the graph G). Therefore $y = v$, then y is adjacent to some vertex p in S_1. Thus, from all cases can say that S_1 is a total dominating set in the given graph G not containing v with $|S_1| = \gamma_T(G)$.Therefore, S_1 is a γ_T-set in the graph G not containing v, a contradiction to the fact that $v \in V_T^+$. Therefore, $TPr\,[v,S]$ contains at least two vertices. Therefore, $TPr\,[v,S]$ contains at least two vertices different from v.

(iii) Let $w_1, w_2 \in TPr[v,S]$ and w_1, w_2 are two adjacent vertices. Suppose$w_1, w_2 \in S$. Therefore, w_1 is adjacent to w_2 and v, both are in S. Therefore, $w_1 \notin TPr[v,S]$, a contradiction to the fact that $w_1 \in TPr[v,S]$. Therefore, the assumption is wrong. Therefore, at least one $w_i \notin S$ for $i = 1,2$.

Theorem 2.4 Let G be any graph and $V_T^i = \emptyset$. If $v \in V_T^+$ and $w \in V_T^-$, then v and w are non-adjacent vertices.

Proof Let G be any graph. Let $v \in V_T^+$ and $w \in V_T^-$. Suppose v and w are adjacent. Since $w \in V_T^-$, there is a minimum total dominating set S in the graph G not containing w and a vertex $z \in S$ different from v such that $TPr[v,S] = \{w\}$ (because if $z = v$ and $v \in V_T^+$ such that $v \notin V_T^i$, then by theorem 2.3(ii), $TPr[v,S]$ contains at least two vertices different from v, a contradiction to the fact that $TPr\,[v,S] = \{w\}$). But $v \in V_T^+$.Therefore $v \in S$.Therefore, w is adjacent to two vertices v and z, both are in S. Therefore $w \notin TPr[z,S]$, a contradiction to the fact that $TPr[z,S] = \{w\}$. Therefore, the assumption is wrong. Therefore, v and w are non-adjacent.

Theorem 2.5 Let G be any graph and $V_T^i = \emptyset$. Then, $|V_T^0| \geq 2|V_T^+|$.

Proof Let G be any graph and $V_T^i = \emptyset$. Let $v \in V_T^+$ and S be a γ_T-set in the graph G containing v. Therefore by theorem 2.3(ii), $TPr\,[v,S]$ contains at least two vertices w_1, w_2 different from v. Since w_1 is adjacent to v, $w_1 \notin V_T^-$ (by theorem 2.4. Similarly say that $w_2 \notin V_T^-$.

Case 1. Suppose $w_1, w_2 \notin S$.Therefore $w_1, w_2 \notin V_T^+$(by theorem 2.1). Therefore $w_1, w_2 \in V_T^0$. Thus, a vertex $v \in V_T^+$ gives rise to at least two vertices of V_T^0.

Case 2. Suppose w_1 or w_2 belongs to S. Without loss of generality, suppose $w_1 \in S$ and $w_2 \notin S$. From the above case, $w_2 \in V_T^0$. If $w_1 \notin V_T^+$, then $w_1 \in V_T^0$. Thus, a vertex of V_T^+ give rise to at least two vertices of V_T^0. If $w_1 \in V_T^+$, then by theorem 2.3(ii), $TPr[w_1,S]$ contains at least two vertices different from w_1 in which one vertex(say) z_1 is different from v and $z_1 \notin S$ (because if $z_1 \in S$, then w_1 is adjacent to two vertices v and z_1, both are in S and therefore $w_1 \notin TPr[v,S]$. Therefore $z_1 \notin V_T^+$(by theorem 2.1) and w_1 is adjacent z_1. It follows that $z_1 \notin V_T^-$(by theorem 2.4). Therefore, $z_1 \in V_T^0$. Also,$w_2 \notin TPr[w_1,S]$ (because if $w_2 \in TPr[w_1,S]$, then w_1 is adjacent to v and w_2(both are in S) and therefore $w_1 \notin TPr[v,S]$, a contradiction to the fact $w_1 \in TPr\,[v,S]$). Therefore, $z_1 \neq w_2$ and $z_1, w_2 \in V_T^0$. Thus, a vertex $v \in V_T^+$ gives rise to two distinct vertices of V_T^0.

Case 3. Let $w_1, w_2 \in S$. If $w_1 \notin V_T^+$ and $w_2 \notin V_T^+$, then $w_1, w_2 \in V_T^0$. Thus, a vertex $v \in V_T^+$ gives rise to two distinct vertices of V_T^0. If $w_1 \in V_T^+$ and $w_2 \notin V_T^+$, then from case 2, say that a vertex $v \in V_T^+$ gives rise to two distinct vertices of V_T^0. If $w_1, w_2 \in V_T^+$, then as per above case, there exists two distinct vertices z_1,z_2 such that $z_i \in TPr[w_i,S]$, for $i = 1,2$ with both $z_1, z_2 \notin S$ (because if $z_i \in S$, then w_i is adjacent to v and z_i, both are in S and therefore $w_i \notin TPr[v,S], i =\,,2$). Hence, $z_1, z_2 \notin V_T^+$ (by theorem 2.1). But w_i is adjacent to z_i and $w_i \in V_T^+$ for $i = 1,2$. Therefore, $z_i \notin V_T^-$, $i = 1,2$ (by theorem 2.4). Therefore, $z_1, z_2 \in V_T^0$.Therefore, a vertex $v \in V_T^+$ gives rise to two distinct vertices of V_T^0. Thus, proved that every vertex $v \in V_T^+$ gives rise to at least two distinct vertices of V_T^0. Suppose v_1 and v_2 are two distinct vertices of V_T^+ such that $x_1, x_2 \in V_T^0$ corresponds to v_1 with respect to a γ_T-set in S in the graph G and $x_3, x_4 \in V_T^0$ corresponds to a vertex v_2 with respect to the same γ_T-set S.

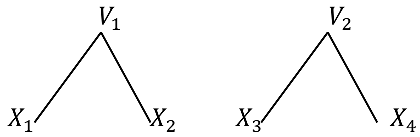

Fig 1

Here the possibility for x_i is either w_i or z_i. Suppose $x_2 = x_3$. Then, x_2 is adjacent to two distinct vertices of S. Therefore, $x_2 \notin TPr[w,S]$ for any $w \in S$. Therefore, $x_2 \notin TPr[w,S]$, for $w = w_1, w_2, v_1, v_2$, a contradiction. Thus, proved that two distinct vertices of V_T^+ gives rise to two distinct two elements sets of V_T^0. Therefore, $|\,V_T^0| \geq 2|\,V_T^+|$.

Theorem 2.6 Let G be any graph and $V_T^i = \emptyset$. If $\gamma_T(G-v) \neq \gamma_T(G)$ for every vertex $v \in V(G)$, then $\gamma_T(G-v) < \gamma_T(G)$ for every $v \in V(G)$.

Proof To prove that $v \in V_T^-$. That is, $V_T^- = V(G)$. Since $\gamma_T(G-v) \neq \gamma_T(G)$ for every $v \in V(G)$, $v \notin V_T^0$ for every $v \in V(G)$. Therefore, $V_T^0 = \emptyset$. Therefore, $|\,V_T^0| = 0$. But $|\,V_T^0| \geq 2|\,V_T^+|$ (by theorem 2.4). Therefore, $|\,V_T^+| = 0$. Therefore, $V_T^+ = \emptyset$. Also given that $V_T^i = \emptyset$. But, $V_T^+ \cup V_T^- \cup V_T^0 \cup V_T^i = V(G)$. Therefore,$V(G) = V_T^-$. Therefore, $\gamma_T(G-v) < \gamma_T(G)$ for every $v \in V(G)$.

3. CONCLUSION

Finally this paper concludes that if the total domination number changes whenever every edge is removed, then the total domination number increases. Accordingly, the total domination number decreases after the removal of any vertex.

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