An Extension Of Nesic's Result For Weakly Compatible Maps

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Abstract. *A result of Nesic is extended to two pairs of self-maps through the notions of weak compatibility and orbital completeness of the metric space.*

2000 AMS mathematics subject classification: 54 H 25
Key Words: Weakly compatible self-maps, orbit and common fixed point

1. Introduction

In 2003, Nesic [2] proved the following Theorem:

Theorem 1: *Let f and g be self-maps on a metric space satisfying the general inequality*

$$[1+ad(x,y)]\, d(fx,gy) \le a[d(x,fx)d(y,gy)+d(x,gy),d(y,fx)]$$
$$+\, b\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}[d(x,gy)+d(y,fx)]\right\}$$
$$\text{for all } x,y\in X, \qquad \ldots \qquad \textbf{(1)}$$

where $a \ge 0$ *and* $0 \le b < 1$.

(i) *If there is a subsequence of the associated sequence* $<x_n>$ *at* x_0 *converging to some* $z \in x$, *where*

$$\left.\begin{aligned} x_{2n-1} &= fx_{2n-2} \\ x_{2n} &= gx_{2n-1} \end{aligned}\right\} \qquad \ldots \qquad \textbf{(2)}$$

then f and g have a unique common fixed point.

In this paper we extend theorem1 to two pairs of weakly compatible maps [1] using the notion orbital completeness of the metric space.

2. Preliminaries:

In this paper (X, d) denotes a metric space and f and g self-maps on it.

Given a pair of self-maps S and T on X, an (f, g) orbit at x_0 relative to (S, T) is defined by

$$\left.\begin{aligned} y_{2n-1} &= fx_{2n-2} = Sx_{2n-1} \\ y_{2n} &= gx_{2n-1} = Tx_{2n},\ n=1,2,3,.... \end{aligned}\right\} \qquad \ldots \qquad \textbf{(3)}$$

provided the sequence $\{y_n\}_{n=1}^{\infty}$ exists [3].

Remark 1: If $S = T = I_X$ the identify map on X we get (2) from (3) as a particular case.

Remark 2: Let $f(x) \subseteq g(x)$ and $g(x) \subseteq T(x)$ … **(4)**

and $x_0 \in X$. Then by induction on n the (f,g) orbit at x_o w.r.t. (S, T) with choice (3) can be defined.

Definition 1: The space X is (f,g) –orbitally complete w.r.t. the pair (S,T) at x_o if every Cauchy sequence in the orbit (3) converges in x.

Remark 3: If $S = T = I_X$ then condition (i) of Theorem 1 follows from orbital completeness.

Definition 2: A point $z \in X$ is a coincidence point of self-maps f and T if $f_z = T_z$, while z is a common coincidence point for pairs (f,T) and (S,g) if $fz = gz = Sz = Tz.$.

Definition 3: Self-maps f and T are said to be weakly compatible [1] if they commute at their coincidence point.

Our Main Result is

Theorem 2: *Let f, g, S and T be self-maps on X satisfying the inclusions (4) and the inequality*

$$[1 + ad(Tx, Sy)]\, d(fx, gy) \le a[d(Tx, fx)d(Sy, gy) + d(Tx, gy), d(Sy, fx)] + b\max\left\{d(Tx, Sy), d(Tx, fx), d(Sy, gy), \frac{1}{2}[d(Tx, gy) + d(Sy, fx)]\right\}$$

$$\text{for all } x, y \in X, \qquad \ldots \qquad \textbf{(5)}$$

where the constants a and b have the same choice as in Theorem 1.
(*ii*) *Given* $x_0 \in X$ *, suppose that X is (f,g) orbitally complete w.r.t. (S,T) at* x_0.
(*iii*) *S and T are onto*
and
(*iv*) *(g,S) and (f,T) are weakly compatible.*
Then the four self-maps will have a common coincidence point, which will also be a unique common fixed point for them.

Proof: Let $x_0 \in X$. By Remark 2, the (f,g) orbit can be described as in (3).
Write $t_n = d(y_n, y_{n-1})$ for $n \ge 1$. Taking $x = x_{2n-2}, y = x_{2n-1}$ in the inequality (5) and using (3),

$$[1 + ad(Tx_{2n-2}, Sx_{2n-1})]\, d(fx_{2n-2}, gx_{2n-1}) \le a[d(Tx_{2n-2}, fx_{2n-2})d(Sx_{2n-1}, gx_{2n-1}) + d(Tx_{2n-2}, gx_{2n-1}), d(Sx_{2n-1}, fx_{2n-2})] + b\max\{d(Tx_{2n-2}, Sx_{2n-1}), d(Tx_{2n-2}, fx_{2n-2}), d(Sx_{2n-1}, gx_{2n-1}), \frac{1}{2}[d(Tx_{2n-2}, gx_{2n-1}) + d(Sx_{2n-1}, fx_{2n-2})]\},$$

$$\Rightarrow \quad [1 + ad(y_{2n-2}, y_{2n-1})]\, d(y_{2n-2}, y_{2n}) \le a[d(y_{2n-2}, y_{2n-1})d(y_{2n-1}, y_{2n}) + d(y_{2n-2}, y_{2n}), d(y_{2n-1}, y_{2n-1})] + b\max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n}), \frac{1}{2}[d(y_{2n-2}, y_{2n}) + d(y_{2n-1}, y_{2n-1})]\}$$

$$\Rightarrow \quad t_{2n-1} \le b\max\{t_{2n-2}, t_{2n-1}\}. \qquad \ldots \qquad \textbf{(6)}$$

Similarly taking $x = x_{2n-2}, y = x_{2n-3}$ in (5) and using (3) and preceding as above we get

$$t_{2n-2} \le b\max\{t_{2n-3}, t_{2n-2}\}. \qquad \ldots \qquad \textbf{(7)}$$

From (6) and (7), we see that

$$t_n \le b\max\{t_{n-1}, t_n\} \text{ for all } n \ge 2. \qquad \ldots \qquad \textbf{(8)}$$

If $\max\{t_{n-1,}t_n\}=t_n$, from (8), $t_n \le bt_n < t_n$ a contradiction, and $\max\{t_{n-1}\}\le t_{n-1} \le t_n = 0 \Rightarrow t_{n-1}=0$

Therefore, $y_{n-1}=y_n=y_{n+1}$ and the inequality (8) holds good.

We take $\max\{t_{n-1,}\, t_n\}=t_{n-1}$ for all *n*.

So that from (8),

$$t_n \le bt_{n-1} \qquad \text{for all } n \ge 2. \qquad \dots \qquad \textbf{(9)}$$

Repeated application of (9) gives

$$t_n \le b^{n-1}t_1 \qquad \text{for all } n \ge 2. \qquad \dots \qquad \textbf{(10)}$$

Now for *m* > *n*, by triangle inequality and (10),

$$d(y_{m,}\, y_n) \le d(y_m, y_{m-1}) + d(y_m, y_{m-2}) + \cdots + d(y_{n+1}, y_n) \quad (m\text{-}n \text{ terms})$$

$$= t_{m-1} + t_{m-2} + ... + t_n \le \left(b^{m-1} + b^{m-2} + ... + b^{n-1}\right)t_1$$

$$= b^{n-1}t_1\left(1 + b + b^2 + ... + b^{m-n}\right) \le b^{n-1}t_1(1 + b + b^2 + ...) = \frac{b^{n-1}t_1}{1-b} \text{ for all } n \ge 1$$

Applying the limit as $m, n \to \infty$ this gives $d(y_n, y_m) \to \infty$, since $\lim\limits_{n\to\infty} b^{n-1} = 0$ as $0 \le b < 1$.

Hence $< y_n >_{n=1}^{\infty}$ is Cauchy sequence in the orbit (3). By orbital completeness of *X*, $\lim\limits_{n\to\infty} y_n = z$ for some $z \in X$. That is

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} fx_{2n-2} = \lim_{n\to\infty} Sx_{2n-1} = z \qquad \dots \qquad \textbf{(11)}$$

and $$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} gx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z\,. \qquad \dots \qquad \textbf{(12)}$$

Since *S* and *T* are onto, $z = Su$ and $z = Tv$ for some $u, v \in X$

we prove that $Su = gu$ and $Tu = fv$.

Put $x = x_{2n-2,} y = u$ in the inequality (5)

$$[1 + ad(Tx_{2n-2}, Su)]\, d(fx_{2n-2}, gu) \le a[d(Tx_{2n-2}, fx_{2n-2})d(Su, gu) + d(tx_{2n-2}, gu), d(Su, fx_{2n-2})]$$
$$+ b\max\{ d(Tx_{2n-2}, Su), d(Tx_{2n-2}, fx_{2n-2}), d(Su, gu),$$
$$\frac{1}{2}[d(Tx_{2n-2}, g_u) + d(s_u, fx_{2n-2})]\}$$

As $n \to \infty$, this implies

$$[1 + ad(z,z)d(z,gu) \le a[d(z,z)\, d\, d(z,gu) + d(z,gu)\, d(su,z)] + b\max\{ d(z,z), d(z,z), d(z,gu)$$
$$+ \frac{1}{2}[d(z,gu) + d(z,z)]\}$$

so that $d(z,gu) \le b\, d(z,gu)$ or $z = gu$. Thus $Su = gu = z$. This and weak compatibility of *g* and *S* implies that $Sgu = gsu$ or $Sz = gz$.

On the other hand, taking $x = v$ and $y = x_{2n-1}$ in (5)

$$[1 + ad(Tv, Sx_{2n-1})]\, d(fv, gx_{2n-1}) \le a[d(Tv, fv)d(Sx_{2n-1}, gx_{2n-1}) + d(Tv, gx_{2n-1})d(Sx_{2n-1}, fv)]$$
$$+ b\max\{ d(Tv, Sx_{2n-1}), d(Tv, fv), d(Sx_{2n-1}, gx_{2n-1}),$$
$$\frac{1}{2}[d(Tv, gx_{2n-1}) + d(Sx_{2n-1}, fv)]\}$$

Applying lim as $n \to \infty$

$$[1 + ad(Tv, z)]\, d(fv, z) \le a[d(Tv, fv)d(z,z) + d(Tv, z)d(z, fv)]$$

$$+b\max\{d(Tv,z),d(Tv,fv),d(z,z),$$
$$\frac{1}{2}[d(Tv,z)+d(z,fv)]\}$$

So that $d(fv,z)\leq bd(z,fv)$ or $fv=z=Tv$. By weak compatibility of (*f*, *T*) we get $fTv=Tfv\Rightarrow fz=Tz$.

Again taking $x=y=z$ in (5)

$$[1+ad(Tv,Sz)]\,d(fz,gz)\leq a[d(Tz,fz)d(Sz,gz)+d(Tz,gz)d(Sz,Tz))]$$
$$+b\max\{d(Tz,Sz),d(Tz,fz),d(Sz,gz),$$
$$\frac{1}{2}[d(Tz,gz)+d(Sz,fz)]\}$$

So that $[1+ad(fz,gz)]d(fz,gz)\leq a[0+d(fz,gz)d(gz,fz)]$

$$+\mathrm{b}\max\left\{\mathrm{d}(\mathrm{fz},\mathrm{gz}),0,0,\frac{1}{2}[\mathrm{d}(\mathrm{fz},\mathrm{gz})+\mathrm{d}(\mathrm{fz},\mathrm{gz})]\right\}$$

Or $d(fz,gz)\leq bd\,(fz,gz))\Rightarrow fz=gz$.
Thus $fz=gz=Tz=Sz$, that is *z* is a common coincidence point of *f*, *g*, *T* and *S*.

Finally writing $x=x_{2n},y=z$ in (5),
.

$$[1+ad(Tx_{2n},Sz)]\,d(fx_{2n},gz)\ \leq a[d(Tx_{2n},fx_{2n})d(Sz,gz)+d(Tx_{2n},gz),d(Sz,fx_{2n})]$$
$$+b\max\{d(Tx_{2n},Sz),d(Tx_{2n},fx_{2n}),d(Sz,gz),$$
$$\frac{1}{2}[d(Tx_{2n},gz)+d(Sz,fx_{2n})]\}.$$

Appling limit as $n\rightarrow\infty$, this gives

$$[1+ad(z,gz)]\,d(z,gz)\ \leq a[d(z,z)d(gz,gz)+d(z,gz),d(gz,z)$$
$$+b\max\{d(z,gz),d(z,z),d(gz,gz),\ \frac{1}{2}[d(z,gz)+d(gz,z)]$$

Or $d(z,gz)\leq b\,d(z,gz)\Rightarrow gz=z$. Hence $fz=gz=Tz=Sz=z$.
Thus *z* is a common fixed point of *f*, *g*, *T* and *S*.
Uniqueness: Let *z*, *z′* be two common fixed points taking $x=z,y=z'$ in (5),

$$[1+ad(Tz,Sz')]\,d(fz,gz')\ \leq a[d(Tz,fz)d(Sz',gz')+d(Tz,gz'),d(Sz',fz)]$$
$$+b\max\{d(Tz,Sz'),d(Tz,fz),d(Sz',gz'),\ \frac{1}{2}[d(Tz,gz')+d(Sz',fz)]\}$$

So that $d(z,z')\leq b(z,z')$ or $z=z'$. Hence the common fixed point is unique.

Remark 4: It is well known that identity map commutes with every self map and hence (*f*, *T*) = (*f*, *I*) and (*g*, *S*) = (*g*, *I*) are weakly compatible pairs. Also *I* is onto.

In view of Remarks 1, 2 and 3, a common fixed point of *f* and *g* is ensured by Theorem 2. Thus Theorem 2 extends Theorem1 significantly.

References:

[1] Gerald Jungck and Rhoades, B.E., *Fixed point for set-valued functions with out continuity*, Indian J. pure appl. Math. **29 (3)** (1998), 227-238.
[2] S.C Nesic , C*ommon fixed point theorems in metric spaces*. Bull. math. Soc. Sci. Math. Roumanie **46(94)** (2003), 149-155.
[3] T. Phaneendra, Coincidence points of two weakly compatible self maps and common fixe point Theorem through orbits, Ind. J .Math. **46 (2-3**) (2004), 173-180.