An Optimal Design of Reliability Test Plans for a Parallel System Using Integer Programming

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Abstract: Consider a problem of acceptance testing for a parallel system with independent components having constant but unknown failure rates. The components are individually tested and the tests are terminated as soon as the pre-assigned numbers of components fail. In this paper, we obtain a reliability test plan based on unbiased estimator of component failure rates which leads to minimum testing cost and satisfying the usual probability requirements. Further, we solve a resulting nonlinear integer optimization problem. Finally, we illustrate our approach through numerical examples.

Keywords: Parallel System, System Reliability, Component testing, Cost minimization, Optimization, Type I and Type II errors, Integer programming.

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1Introduction

Manufacturers of the products want to test their products for reliability to make sure that it lasts for longtime. They need simple and cost effective technique to test the reliability. Rajgopal and Mazumdar (1996) obtained reliability test plans for series system based on Type II censoring. This work led to the conclusion that all components has to be tested equally. Rajagopal and Mazumdar (1988) examined component testing procedure that provides a criterion for accepting or rejecting the parallel system based on sum of the logarithms on the total times on test for each component and it needs several approximation procedures for obtaining the parameters for the test plan, and in general, the methods discussed were difficult for implementation for testing the system with more than two components. Emre Yamangil et al. (2011) discuss the system reliability test plans by considering the system performance measures, namely, MTTF and system availability concepts. However, the problem of testing the reliability of parallel system with more than two components remained complicated under the assumptions mentioned in Section 2.

In this paper, we consider a parallel system consisting of n independent components with constant and unknown failure rates. Under the assumptions made in Section 2, we obtain the data from Type- II censoring, that is, a predetermined number of each type of component is tested to failure with replacement after each failure. We describe a procedure to develop an optimal reliability test plan based on the unbiased estimator of failure rates of components. These procedures are simpler than thosediscussed in Rajgopal and Mazumdar (1988), and it can be applied to a general n component parallel system. We obtain the optimal reliability test plans by solving a nonlinear integer optimization problem. Finally, we design a general algorithm for n component parallel system and implement it in LINGO software which will give the optimal value of the test plan. Further, we illustrate our procedure for a 3–component parallel system.

This paper is organized as follows. In Section 2, we give a set of assumptions needed to develop the solution of the problem. In Section 3, we formulate the problem and find an unbiased estimator of failure rate and hence the estimate for system reliability. We use well-known delta method (see, Rajgopal and Mazumdar (1997)) for obtaining the distribution of test statistic. In Section 4, we give algorithm to solve non-linear integer optimization problem and discuss some numerical examples to illustrate our approach. In Section 5, we give conclusions and discuss the scope for further research.

2. Assumptions

We make the following assumptions before the problem formulation.

a. The system representation has n independent components in parallel.
b. The failure rate of the component i is exponentially distributed random variable with unknown constant failure rate λ_i, $1 \le i \le n$.
c. During the testing, whenever a component fails, we replace it immediately by an *i.i.d.* component.

d. The interfaces within the system are perfect and no need to test them and the system is highly reliable.
e. The failure rate of each individual component is small, so that the system reliability *R(t)* for *t* units of time, can be approximated by

$$R(t) = 1 - \prod_{i=1}^{n} \lambda_i t.$$

Under the above assumptions, the expression for reliability estimate of the parallel system with *n* independent components , for a mission time of one unit, is given by

$$\hat{R}_s = 1 - \prod_{i=1}^{n} \hat{\lambda}_i,$$

where $\hat{\lambda}_i$ is an unbiased estimate of failure rate to be obtained later.

Note that Assumptions (*d*) and (*e*), can be seen in Rajgopal and Mazumdar (1997).

3. Problem Formation

A. An unbiased estimator of failure rates.

We test the *i*-th component for k_i number of times. Let X_i denote lifetime of the *i*-th component. Let T_i be defined by

$$T_i = \sum_{i=1}^{k_i} X_i,$$

which is the total of k_i failure times. Then

if $X_i \sim \exp(\lambda_i)$, $T_i \sim G(k_i, \lambda_i)$.

We see that

$$E(\frac{1}{T_i}) = \frac{\lambda_i}{(k_i - 1)}$$

$$\lambda_i = E\left(\frac{k_i - 1}{T_i}\right).$$

$$Hence \quad \hat{\lambda}_i = \frac{(k_i - 1)}{T_i}, 1 \le i \le n.$$

Thus $\hat{\lambda}_i$ is an unbiased estimator of λ_i, $1 \le i \le n$.

Suppose that c_i is the cost of testing the i-th component for unit time period, then our objective is to solve the following non-linear optimization Problem P:

$\min \sum_{i=1}^{n} c_i k_i$ such that

Pr {accept system | $R(1) \le R_0$ } $\le \alpha$,

Pr {reject system |$R(1) \ge R_1$ } $\le \beta$,

where $R(1)$ is the reliability of the system for unit time, and 0 <R_0<R_1< 1, $\alpha+\beta$< 1. Here we call $R_O$as lower bound probability for rejecting the system, and R_1 the lower bound probability for accepting the system. That is, when $R(1) \le R_0$we reject the system and in this case, the system is bad. Also when $R(1) \ge R_1$ we accept the system and in this case, the system is good.

The test procedure is to "Rejectthe system if $\hat{R}_s < B$, and accept otherwise", where$B$ is the constant to be determined later and $\hat{R}_s = 1 - \prod_{i=1}^{n} \hat{\lambda}_i$.

We have the following two cases

Case1. $\hat{R}_s \le B$. Here we have

$$1 - \prod_{i=1}^{n} \left(\frac{k_i - 1}{T_i} \right) \le B$$

That is *iff* $\prod_{i=1}^{n} \left(\frac{k_i - 1}{T_i} \right) \ge d$.

Case2. $\hat{R}_s \ge B$. *Then we have*

$$\prod_{i=1}^{n} \left(\frac{k_i - 1}{T_i} \right) \le d \text{ , where } 1 - B = d.$$

Since the system is highly reliable, the acceptance rule becomes $\prod_{i=1}^{n} \left(\frac{k_i - 1}{T_i} \right) \le d$, where $d \varepsilon (0,1)$.

Also note that

$$\prod_{i=1}^{n} \left(\frac{k_i - 1}{T_i} \right) \le d \Leftrightarrow \ln \left[\prod_{i=1}^{n} \left(\frac{k_i - 1}{T_i} \right) \right] \le \ln d$$

$$\Leftrightarrow \sum_{i=1}^{n} \ln \left(\frac{k_i - 1}{T_i} \right) \le M, \ where \ M = \ln d.$$

That is,

$$T=\sum_{i=1}^{n}\ln\left(\frac{k_i-1}{T_i}\right)\le M.$$

(3.1)

This gives the alternate rule of acceptance of the system. The value of M will be determined in next section.

B. Distribution of the test statistics using delta method

We have $T_i \approx N(\frac{k_i}{\lambda_i},\frac{k_i}{\lambda_i^2})$.

Let $g(T_i)=\ln(\frac{k_i-1}{T_i})$, $T_i \approx N(\frac{k_i}{\lambda_i},\frac{k_i}{\lambda_i^2})$. Let $\mu=\frac{k_i}{\lambda_i}$. Then by well-known delta method, we have

$g(T_i)\approx N(g(\mu),(g^{'}(\mu))^2V(T_i))$,

$g^{'}(T_i)=-\frac{1}{T_i}$. Thus $g^{'}(\mu)=-\frac{\lambda_i}{k_i}$.

and

$(g^{'}(\mu))^2V(T_i)=\frac{1}{k_i}$.

Now $g(T_i)\approx N\left[\ln\left(\frac{\lambda_i(k_i-1)}{k_i}\right),\frac{1}{k_i}\right]$.

Using Lindebergcentral limit theorem, we have

$$g(T)=\sum_{i=1}^{n}g(T_i)\sim N\left[\sum_{i=1}^{n}\ln\left(\frac{\lambda_i(k_i-1)}{k_i}\right),\sum_{i=1}^{n}\frac{1}{k_i}\right],$$ and

$$Z=\frac{g(T)-\mu_T}{\sigma_T^2}\sim N(0,1),\ where\ \mu_T=\sum\ln\left(\frac{\lambda_i(k_i-1)}{k_i}\right)\ and\ \sigma_T^2=\sum\frac{1}{k_i}.$$

C. Solution of the problem P

Let*Z~N(0,1)*be standard normal random variable. Thenusing the rule stated above in equation (3.1), the probability of acceptance of the system becomes

Pr {accept the system | $R(1) \le R_0$ } $\le \alpha$

$$\Pr\left\{ Z \le \frac{M - \mu_T}{\sigma_T} \,|\, 1 - \prod_{i=1}^{n} \lambda_i \le R_o \right\} \le \alpha$$

$$= \Pr\left\{ Z \le \frac{M - \mu_T}{\sigma_T} \,|\, \sum_{i=1}^{n} \ln \lambda_i \ge \ln(1 - R_0) \right\} \le \alpha,$$

and

Pr {reject the system | $R \ge R_1$ } $\le \beta$ which is equivalent to

Pr {accept the system | $R(1) \ge R_1$ } $\ge 1 - \beta$,
becomes

$$\Pr\left\{ Z \le \frac{M - \mu_T}{\sigma_T} \,|\, \sum_{i=1}^{n} \ln \lambda_i \le \ln(1 - R_1) \right\} \le 1 - \beta.$$

Then the Problem P becomes Problem P1 as under:

$$\min_{\lambda_i} \sum_{i=1}^{n} c_i k_i$$

$$\max_{\lambda_i} \Pr\left\{ Z \le \frac{M - \mu_T}{\sigma_T} \,|\, \sum_{i=1}^{n} \ln \lambda_i \ge \ln(1 - R_0) \right\} \le \alpha$$

$$\min_{\lambda_i} \Pr\left\{ Z \le \frac{M - \mu_T}{\sigma_T} \,|\, \sum_{i=1}^{n} \ln \lambda_i \le \ln(1 - R_1) \right\} \le 1 - \beta.$$

Let $\phi(z)$ be the cumulative distribution function of the standard normal random variable.Note that it is strictly increasing function in its argument. So the Problem P1 can be rewritten as Problem P2 as

$$\min_{\lambda_i} \sum_{i=1}^{n} c_i k_i$$

such that

$$\max_{\lambda_i}\left(\frac{M-\mu_T}{\sigma_T} \mid \sum_{i=1}^{n}\ln\lambda_i \geq \ln(1-R_0)\right) \leq Z_\alpha$$

$$\min_{\lambda_i}\left(\frac{M-\mu_T}{\sigma_T} \mid \sum_{i=1}^{n}\ln\lambda_i \leq \ln(1-R_1)\right) \geq Z_{1-\beta}.$$

After substituting for μ_T and σ_T, the Problem P2 becomes Problem P3:

$$\min_{\lambda_i} \sum_{i=1}^{n} c_i k_i$$

such that

$$\max_{\lambda_i}\left(\frac{M-\sum_{i=1}^{n}\ln\left(\frac{\lambda_i(k_i-1)}{k_i}\right)}{\sqrt{\sum_{i=1}^{n} 1/k_i}} \mid \sum_{i=1}^{n}\ln\lambda_i \geq \ln(1-R_0)\right) \leq Z_\alpha$$

$$\min_{\lambda_i}\left(\frac{M-\sum_{i=1}^{n}\ln\left(\frac{\lambda_i(k_i-1)}{k_i}\right)}{\sqrt{\sum_{i=1}^{n} 1/k_i}} \mid \sum_{i=1}^{n}\ln\lambda_i \leq \ln(1-R_1)\right) \geq Z_{1-\beta}.$$

4. An Algorithm to solve Problem P3

A. An algorithm

1. Initialize $j = 0, n = n_0, R_0, R_1, \alpha, \beta, d \in (0,1)$.

2. Initialize $\lambda_i^{(j)}$ such that $\prod_{i=1}^{n} \lambda_i^{(j)} \geq 1 - R_0$ in

$$L_1 = \left[\frac{\ln d - \sum_{i=1}^{n} \ln(\frac{\lambda_i^{(j)}(k_i^{(j)}-1)}{k_i^{(j)}})}{\sqrt{\sum_{i=1}^{n} 1/k_i^{(j)}}} \,|\, \prod_{i=1}^{n} \lambda_i^{(j)} \geq 1 - R_0 \right] \leq Z_\alpha$$

and $\prod_{i=1}^{n} \lambda_i^{(j)} \leq 1 - R_1$ in

$$L_2 = \left[\frac{\ln d - \sum_{i=1}^{n} \ln(\frac{\lambda_i^{(j)}(k_i^{(j)}-1)}{k_i^{(j)}})}{\sqrt{\sum_{i=1}^{n} 1/k_i^{(j)}}} \,|\, \prod_{i=1}^{n} \lambda_i^{(j)} \leq 1 - R_1 \right] \geq Z_{1-\beta}$$

3. If the problem is feasible, find the value of $k_i^{(j)}$ and calculate $C^{(j)} = \sum_{i=1}^{n} c_i k_i^{(j)}$, then go to Step4. Else $j = j + 1$ and go to Step 2.
4. Compare the cost $C^{(j)} < C^{(j+1)}$ then declare $C^{(j)}$ as the minimum cost and go to Step 5.

Otherwise go to Step 2 and repeat the procedure.

5. Stop the algorithm.

B. Numerical Examples

To illustrate the above algorithm, we consider a 3-component parallel system. In Table 1, the first and second column give various values of R_0, and . R_1. The Z values are obtained from statistical table.The third column gives the number of times each component to be tested. The last column gives the minimum cost. Here we are considering the initial cost vector as C= (120,180,150). The value of d chosen in (0, 1) was 0.3.

Table 1

R_0	R_1	k_1, k_2, k_3	Minimum total cost($)
0.82	0.90	2,3,3	1230
0.86	0.92	2,2,2	0900
0.87	0.93	3,3,4	1500
0.95	0.99	3,2,3	1170

From Table 1, we conclude that to achieve the reliability of 0.93, one has to test the first component three times, second component three times and the third component four times, and this plan incurs the minimum cost of $ 1500. Note that in case of series system, one has to test all components equally under Type II censoring(Rajgopal and Mazumdar (1996)) But, here we see that components need not be tested equally.

Table 2

R_0	R_1	k_1, k_2, k_3	Minimum total cost($)
0.80	0.90	3,3,2	34
0.81	0.95	3,2,2	30
0.88	0.99	2,2,2	28

In Table 2, we have taken the initial cost C = (2, 4, 8). We tested for various values of R_0 and R_1. To get a reliability of 0.90, one has to test the first and second components three times, third component only two times.The value of *d* chosen in (0,1) was 0.025. The cost incurred is $ 34.

Table 3

R_0	R_1	k_1, k_2, k_3, k_4	*C= (100,120,160,180)* *Minimum total cost ($)*
0.80	*0.95*	*5,4,3,3*	*2000*
0.82	*0.97*	*5,4,4,4*	*2340*
0.82	*0.98*	*5,4,4,4*	*2340*

Table 4

R_0	R_1	k_1, k_2, k_3, k_4	*C= (100,120,160,180)* *Minimum total cost ($)*
0.88	*0.99*	*3,3,3,2*	*1500*
0.85	*0.98*	*4,2,2,2*	*1320*
0.87	*0.99*	*4,3,2,2*	*1440*

In Table 3, d =0.3 and α=0.05, and in Table 4, d=0.3 α= 0.01.

5. Conclusions and scope for further research

In this work, we have considered a parallel system consisting of n independent components with constant and unknown failure rates. In previous works in the literature, the reliability estimates are not based on unbiased estimators, except for few works. Also in this case, problem was difficult to solve for parallel system with more than 2 components. For a 3-component system, we have solved a nonlinear integer optimization problem using LINGO software. We note here that the number of components of each type to be tested is not equal when we have data from Type II censoring. Also the algorithm developed can be applied for any general *n*- component system. We have made an attempt to design a reliability test plan for a parallel system, which uses unbiased estimator for estimating the failure rate of components. This is essential part in reliability testing. However, there is a wide scope for developments in this area. For example, one can try to work with Weibull distribution as underlying distribution for lifetimes, where one can assume both parameters unknown and can try to estimate them from data.

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