Published by :
http://www.ijert.org

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 11, November-2016

Analysis and Simulation of Slender Curved Beams

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***Abstract*—Curved beams have a wide variety of applications that include switches, clamps, suspensions, tools and other devices. However, the existing formulas of stress and deflection calculations on beams are commonly for straight beams that undergo small linear deflections. They are not applicable to curved beams. The depth of the cross section of a slender curved beam is small compared with its radius of curvature. It usually has large deformation. The load-deflection relationship of a slender curved beam is often nonlinear. It is much more challenging to analyze the deformation of a nonlinear slender curved beam than a linear straight beam. In this paper, the stress calculation formula is presented for slender curved beams. The nonlinear deformations of slender curved beams are analyzed. The deformations and stresses of slender curved beams are simulated. The results of this paper provide a useful roadmap for analyzing and designing slender curved beams.**

Keywords— Curved Beam; Slender Beam; Large Deformation; Stress Analysis; Simulation.

I. INTRODUCTION

Curved beams have a wide variety of applications that include switches, clamps, suspensions, tools and many others [1-2]. However, the stress and deflection calculation formulas on beams in many textbooks of material mechanics are commonly for straight beams that undergo small linear deflections [3].

Figure 1 shows a straight uniform beam with rectangular cross section that is under pure bending. The in-plane depth and out-of-plane width of the cantilever beam are t and b, respectively. The x axis is along the centroidal axis of the straight beam. The neutral axis of the straight beam coincides with its centroidal axis. The normal stress on the cross section from bending is along x direction and can be calculated from the following equation [4-5].

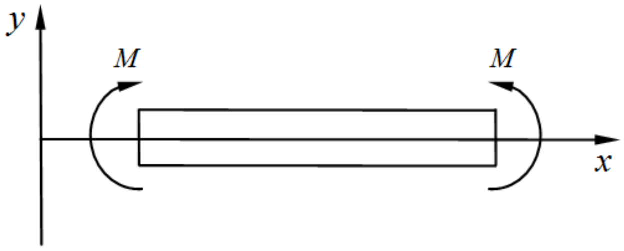

Fig. 1 An initially straight beam under bending.

$$\sigma_x = -\frac{My}{I} \tag{1}$$

I in Equation (1) is the moment of inertia of the cross section, which is $I = \frac{bt^3}{12}$ for rectangular cross section. M is the bending moment. y is the distance of the stress element away from the neutral axis. This equation is often referred to as flexure formula.

Figure 2 shows an initially curved beam with rectangular cross section that is under bending. Plane assumption holds for curved beams, i.e., planar cross sections before bending remain planar after bending. The dash-dot line is the neutral axis of the curved beam. The normal stress on the cross section from bending can be analysed as follows [4-5].

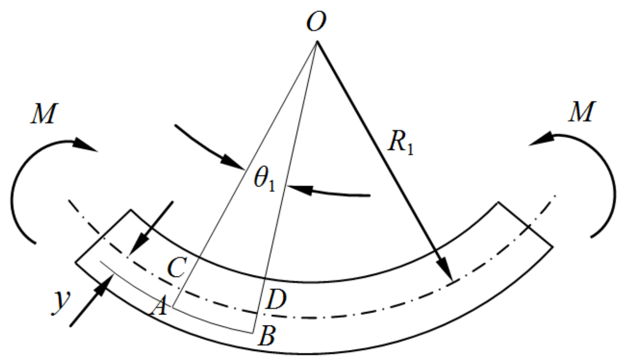

Fig. 2 An initially curved beam under bending.

As shown in Figure 2, element AB is at a distance of y from the neutral axis. The angle formed by AB and its curvature center O of the curved beam element is θ_1 before bending. The angle changes to θ_2 after bending. The element on the neutral axis corresponding to AB is element CD. Because of plane assumption, element CD has the same initial and final angles as element AB. Element CD has initial and final radii of curvature of R_1 and R_2, respectively, on the neutral axis.

The strain of element AB can be derived [6].

$$\varepsilon_{AB} = \frac{(R_2 + y)\theta_2 - (R_1 + y)\theta_1}{(R_1 + y)\theta_1} \tag{2}$$

The arc length of element CD is $R_1\theta_1$ and $R_2\theta_2$, respectively, before and after bending. Since CD is on the neutral axis, it does not change its length during bending. We have $R_1\theta_1 = R_2\theta_2$. Equation (2) can be simplified based on this relationship.

International Journal of Engineering Research & Technology (IJERT)
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$$\varepsilon_{AB} = \frac{y(\theta_2 - \theta_1)}{(R_1 + y)\theta_1} \tag{3}$$

Substituting $\theta_2 = R_1\theta_1/R_2$ into Equation (3) yields the following equation.

$$\varepsilon_{AB} = \frac{y(R_1 - R_2)}{R_2(R_1 + y)} \tag{4}$$

The bending stress of element *AB* is normal to the cross section and can be calculated from ε_{AB}.

$$\sigma = E\frac{y(R_1 - R_2)}{R_2(R_1 + y)} \tag{5}$$

E in Equation (5) is Young's modulus of the beam material. The stress distribution described in Equation (5) that is for a cross section of a curved beam is nonlinear, which makes the neutral and centroidal axes no longer coincide.

The force resultant from the normal stress of pure bending on a cross section of a curved beam is zero, which leads to the following equation.

$$\int_A \sigma dA = \int_A E\frac{y(R_1 - R_2)}{R_2(R_1 + y)}dA = \frac{E(R_1 - R_2)}{R_2}\int_A \frac{y}{(R_1 + y)}dA = 0 \tag{6}$$

Let $r = R_1 + y$. Here *r* references the location of element *AB* from its curvature center. Substituting *r* equation into Equation (6) yields the following equation.

$$\int_A \frac{y}{(R_1 + y)}dA = \int_A \frac{r - R_1}{r}dA = \int_A dA - R_1\int_A \frac{dA}{r} = 0 \tag{7}$$

From Equation (7), the location of the neutral axis can be solved.

$$R_1 = \frac{A}{\int_A \frac{dA}{r}} \tag{8}$$

A is area of the cross section. For a rectangular cross section, we have $A = b(r_2 - r_1) = bt$, $\int_A \frac{dA}{r} = b\ln(r_2/r_1)$ and $R_1 = \frac{(r_2 - r_1)}{\ln(r_2/r_1)}$. Here r_1 and r_2 are the minimum and maximum radii from the curvature center of the rectangular cross section.

The bending moment *M* on the cross section can be derived from the bending stress.

$$M = \int_A \sigma y dA = \frac{E(R_1 - R_2)}{R_2}\int_A \frac{y^2}{(R_1 + y)}dA \tag{9}$$

Substituting $r = R_1 + y$ and $y = r - R_1$ into Equation (9) yields the following equation.

$$M = \frac{E(R_1 - R_2)}{R_2}\left(\int_A r dA - 2R_1\int_A dA + R_1^2\int_A \frac{dA}{r}\right) \tag{10}$$

In Equation (10), $\int_A r dA = A\overline{R}_1$. Here $\overline{R}_1$ references the location of the centroidal axis from its curvature center. Substituting Equation (8) and $\overline{R}_1$ into Equation (10) leads to the following *M* equation.

$$M = \frac{EA(R_1 - R_2)}{R_2}\left(\overline{R}_1 - R_1\right) \tag{11}$$

Substituting Equation (11) into Equation (5) yields the calculation formula on the bending stress.

$$\sigma = \frac{M(r - R_1)}{Ar(\overline{R}_1 - R_1)} \tag{12}$$

Let $e = \overline{R}_1 - R_1$. Here *e* is the distance between the centroidal and neutral axes. Substituting *e*, $r = R_1 + y$ and $y = r - R_1$ into Equation (12) yields the following stress calculation equation.

$$\sigma = \frac{My}{Ae(R_1 + y)} \tag{13}$$

Equations (12) and (13) represent the two forms of the stress calculation formula for curved beams [4-5]. The stress distribution on the cross section is hyperbolic.

From Equation (11), the change in curvature of the neutral axis before and after bending can be determined.

$$\frac{1}{R_2} - \frac{1}{R_1} = \frac{M}{EAeR_1} \tag{14}$$

In order to solve the bending stress by using equation (12) or (13), the bending moment (*M*) on a cross section has to be known. For curved beams that are mainly used for loading bearing such as crane hooks [6], *M* can be calculated based on initially curved shape or undeformed shape since deformation is small. However, slender curved beams often experience large deformation. It is inaccurate to use undeformed shape to calculate bending moment on a cross section. To have accurate stress and deformation analysis results, deformed shape has to be used for bending moment calculation. This makes it challenging to analyse slender curved beams.

The in-plane depth (*t*) is usually much smaller than the radius of curvature (R_1 and R_2) of the neutral axis in slender curved beams. In Equation (4), the *y* part in $R_1 + y$ can be reasonably neglected [6], which leads to the following strain equation.

$$\varepsilon = \frac{y(R_1 - R_2)}{R_2 R_1} = y\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \tag{15}$$

The bending stress of slender curved beams can then be derived from Equation (15) as follows.

$$\sigma = E\varepsilon = E\frac{y(R_1 - R_2)}{R_2 R_1} = Ey\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \tag{16}$$

The stress distribution on the cross section of slender curved beams is considered as linear as shown in Equation (16). The force resultant from the normal stress of pure bending on a cross section of a slender curved beam is zero, which leads to the following equation.

$$\int_A \sigma dA = \int_A E\frac{y(R_1-R_2)}{R_2R_1}dA = \frac{E(R_1-R_2)}{R_2R_1}\int_A ydA = 0 \tag{17}$$

Equation (17) shows that the neutral and centroidal axes in slender curved beams coincide. The bending moment M on the cross section of slender curved beams can be derived from the bending stress.

$$M = \int_A \sigma ydA = \frac{E(R_1-R_2)}{R_2R_1}\int_A y^2dA = EI\frac{(R_1-R_2)}{R_2R_1} \tag{18}$$

Rearranging Equation (18) yields the following equation.

$$\frac{1}{R_2}-\frac{1}{R_1}=\frac{M}{EI} \tag{19}$$

The deformation or deformed shape of slender curved beams can be analysed based on Equation (19) in which the bending moment (M) has to be calculated from the deformed shape.

The material of the analysed slender curved beams in the paper is considered as homogeneous and isotropic. Although slender curved beams undergo large deformations, their strains (ε) remain small and are within the range of elastic deformation. Slender curved beams are analysed in the paper as Euler-Bernoulli curved beams for which plane hypothesis holds. Because of the thin depth and beam flexibility, slender curved beams are considered to be inextensible in the paper during their deformation.

Although it is difficult to solve the large deformation of a slender curved beam from Equation (19) analytically, many different numerical approaches have been proposed and published [7-12]. When R_1 in Equation (19) approaches infinity, a slender curved beam degenerates to a slender straight beam. There are more publications on solving large deformations of slender straight beams [13-19].

Among the existing published papers on slender curved beams, most are focused on solving large deformations under given loadings that include concentrated, distributed or combined. In this paper, the analysed slender curved beams are under given input displacements. The corresponding large deformations and required input forces of analysed slender curved beams are to be solved. The authors of this paper are motived by the challenges facing slender curved beams. The research objective of this paper is to establish an approach and provide a guideline for analysing slender curved beams.

The remainder of the paper is organized as follows. The deformation analysis of slender curved beams is presented in section II. The simulations of deformations and stresses on slender curved beams are provided in section III. Section IV is on the shape design of slender curved beams. Conclusions are drawn in section V.

II. DEFORMATION ANALYSIS OF SLNDER CURVED BEAMS

For the slender curved beam shown in Figure 3, the dashed curve is the undeformed shape of the beam that has a constant radius of curvature of R_1. The solid curve is the deformed shape of the beam that has radius of curvature of R_2 that may change along the curve. The left end of the curved beam is fixed. We assume the displacements of the right free end are given together with the parameters of t and b of the rectangular cross section of the curved beam and Young's modus (E) of the beam material. The required input forces of F_x and F_y at the right end are to be determined in order to generate the desired displacements.

Let the coordinates of an arbitrary point on the solid curve be (x, y). The arc length at the point is s. s meets the condition of $0 \le s \le L$. L here is the total arc length of the curved beam that is considered as inextensible during its deformation. $s = 0$ at the fixed end while $s = L$ at the free end.

The bending moment at an arbitrary point on the solid curve can be derived.

$$M(s) = F_y(x_B - x) - F_x(y_B - y) \tag{20}$$

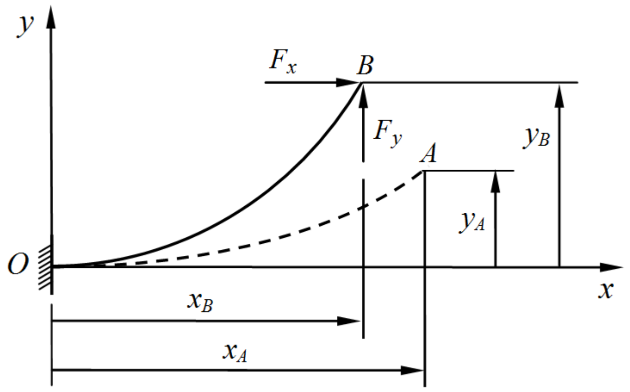

Fig. 3 An analyzed slender curved beam.

Let $\varphi(s)$ be the angle that is from the positive x direction to the tangent line at a point on the solid curve. Then we have the following equation.

$$\frac{d\varphi(s)}{ds}=\frac{1}{R_2(s)} \tag{21}$$

Substituting Equations (20) and (21) into Equation (19) leads to the following equation.

$$EI\frac{d\varphi(s)}{ds}-\frac{EI}{R_1}=F_y(x_B-x)-F_x(y_B-y) \tag{22}$$

Differentiating both sides of Equation (22) with respect to s leads to the following equation.

$$EI\frac{d^2\varphi}{ds^2}=-F_y\frac{dx}{ds}+F_x\frac{dy}{ds} \tag{23}$$

For the solid curve, we have $dx/ds=\cos\varphi$ and $dy/ds=\sin\varphi$. Substituting the expressions of dx/ds and dy/ds into Equation (23) and moving the terms on the right hand side to the left yields the following equation.

$$EI\frac{d^2\varphi}{ds^2}+F_y\cos\varphi-F_x\sin\varphi=0 \tag{24}$$

From Equation (24), we have

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$$\frac{d}{ds}\left[\frac{1}{2}EI\left(\frac{d\varphi}{ds}\right)^2+F_y\sin\varphi+F_x\cos\varphi\right]=0 \qquad (25)$$

Equation (25) results in the following equation.

$$\frac{1}{2}EI\left(\frac{d\varphi}{ds}\right)^2+F_y\sin\varphi+F_x\cos\varphi=C \qquad (26)$$

C in Equation (26) is an arbitrary constant. It can be decided by one boundary condition of the deformed curved beam. At the right end of the solid curve, we have $s=L$, $M(L)=0$, and $\left.\frac{d\varphi}{ds}\right|_{s=L}=0$. Assume the tangent angle of the solid curve at its right end is φ_m. Substituting the expressions of $d\varphi/ds$ and φ at the right end into Equation (26) yields the following equation.

$$F_y\sin\varphi_m+F_x\cos\varphi_m=C \qquad (27)$$

Substituting Equation (27) into Equation (26), we have

$$\left(\frac{d\varphi}{ds}\right)^2=\frac{2}{EI}\left[F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)\right] \qquad (28)$$

Taking square root on both sides of Equation (28) yields the following equation.

$$\frac{d\varphi}{ds}=\sqrt{\frac{2}{EI}}\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)} \qquad (29)$$

Rearranging Equation (29) yields the following equation.

$$ds=\sqrt{\frac{EI}{2}}\frac{d\varphi}{\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)}} \qquad (30)$$

Integrating Equation (30) from the left end to the right end of the solid curve yields the following equation.

$$L=\sqrt{\frac{EI}{2}}\int_0^{\varphi_m}\frac{d\varphi}{\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)}} \qquad (31)$$

Substituting Equation (30) into $dx=\cos\varphi\, ds$ yields the following equation.

$$dx=\sqrt{\frac{EI}{2}}\frac{\cos\varphi\, d\varphi}{\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)}} \qquad (32)$$

Integrating Equation (32) from the left end to the right end of the solid curve yields the following equation.

$$x_B=\sqrt{\frac{EI}{2}}\int_0^{\varphi_m}\frac{\cos\varphi\, d\varphi}{\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)}} \qquad (33)$$

Substituting Equation (30) into $dy=\sin\varphi\, ds$ yields the following equation.

$$dy=\sqrt{\frac{EI}{2}}\frac{\sin\varphi\, d\varphi}{\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)}} \qquad (34)$$

Integrating Equation (34) from the left end to the right end of the solid curve yields the following equation.

$$y_B=\sqrt{\frac{EI}{2}}\int_0^{\varphi_m}\frac{\sin\varphi\, d\varphi}{\sqrt{F_y(\sin\varphi_m-\sin\varphi)+F_x(\cos\varphi_m-\cos\varphi)}} \qquad (35)$$

φ_m, F_y, and F_x are the three unknowns here. They can be solved numerically by Equations (31), (33) and (35).

III. DEFORMATION AND STRESS SIMULATIONS

The deformation and stress of a slender curved beam can be directly analyzed and simulated by finite element analysis (FEA) software ANSYS [20-22]. With the specified input displacements, the required input forces can also be directly obtained from ANSYS through ANSYS simulation.

Figure 4 shows a slender curved beam with the shape of half a circle. The material of the beam is structural steel with Young's modules (E) of 2000 GPa, Poisson's ratio (v) of 0.3, yield strength (σ_y) of 350 MPa. The diameter of the beam is 150 mm. The depth (t) and width (b) of the rectangular cross section of the beam are 0.25 mm and 15 mm, respectively. The left end of the beam is fixed at the origin O of the coordinate system. The right free end A of the beam is on the x axis when the curved beam is undeformed. When A is displaced to A_1, A_2, A_3 and A_4, the deformed beam shape, the stress distribution and the input forces at A are to be determined by simulation in ANSYS. As shown in Figure 4, $A_1A_2A_3A_4$ forms a square with its center at the un-displaced point A and its length of 60 mm.

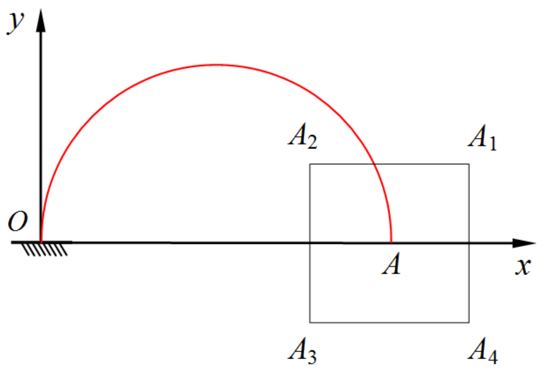

Fig. 4 A slender curved beam with half a circle.

To analyse the curved beam, the solid model of the curved beam is first created in the Design Modeler [23] of ANSYS. ANSYS Design Modeler is an ANSYS Workbench application that provides modeling tool. The solid model created in ANSYS Design Modeler is then meshed and analysed in ANSYS Mechanical [24] that is also an application of ANSYS Workbench.

Figure 5 shows the meshing model of the analysed curved beam. When A is displaced to A_1, the deformed shape of the beam is shown in Figure 6. Because of the thin depth of the curved beam, the deformed shape line is very light in the figure. The colour map and numbers in the figure represent the directional deformation along the x direction. The maximum stress in the deformed beam is 114.61 MPa that is shown in Figure 7. To generate the displacements at A, the required input forces (F_x and F_y) are 0.42261 N and -0.01739 N, respectively. These input forces are the reaction forces in ANSYS.

When A is displaced to A_2, the deformed shape of the beam is shown in Figure 8. The maximum stress in the deformed beam is 246.69 MPa that is shown in Figure 9. The input forces (F_x and F_y) are -0.45321 N and 0.20784 N, respectively. The maximum stress in the deformed beam at A_2 is more than doubled than that at A_1.

When A is displaced to A_3, the deformed shape of the beam is shown in Figure 10. The maximum stress in the deformed beam is 71.098 MPa that is shown in Figure 11. The input forces (F_x and F_y) are -0.11983 N and 0.00818 N, respectively.

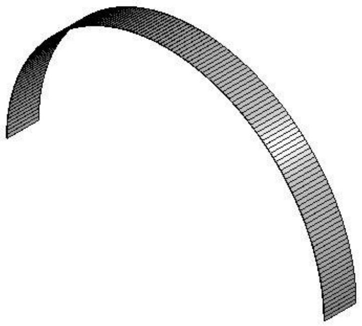

Fig. 5 The meshing model of the curved beam.

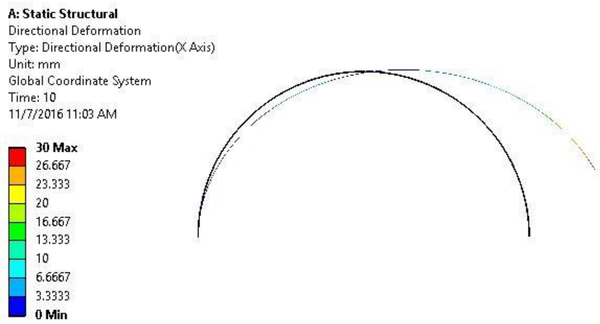

Fig. 6 The deformation of the curved beam at A_1.

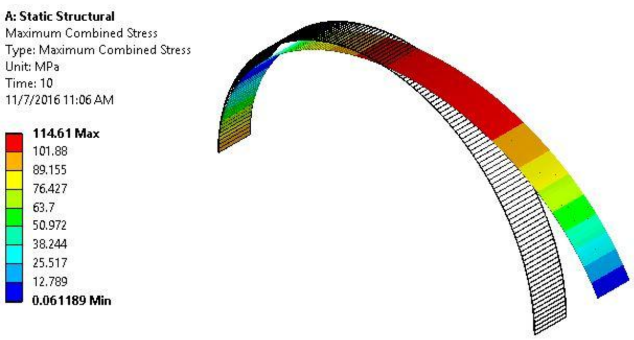

Fig. 7 The stress of the curved beam at A_1.

When A is displaced to A_4, the deformed shape of the beam is shown in Figure 12. The maximum stress in the deformed beam is 363.06 MPa that is shown in Figure 13. The input forces (F_x and F_y) are 0.85707 N and -0.48320 N, respectively. The maximum stress in the deformed beam at A_4 is the highest among all four deformed shapes of the curved beam, which is beyond that of the yield strength of the beam material.

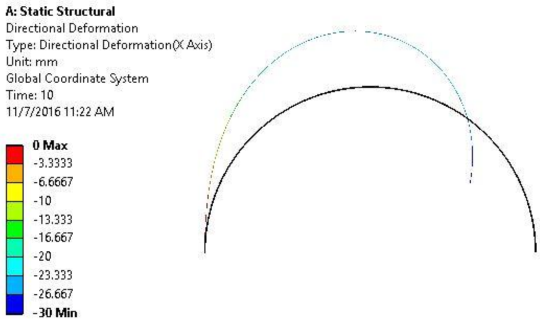

Fig. 8 The deformation of the curved beam at A_2.

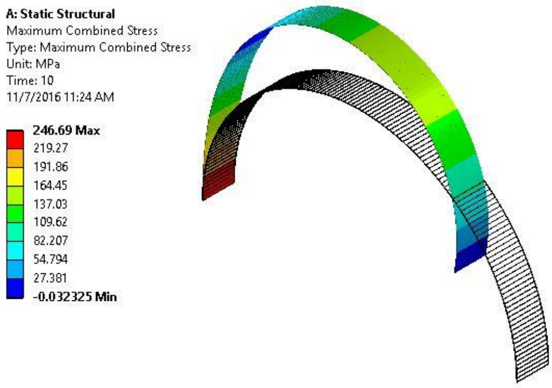

Fig. 9 The stress of the curved beam at A_2.

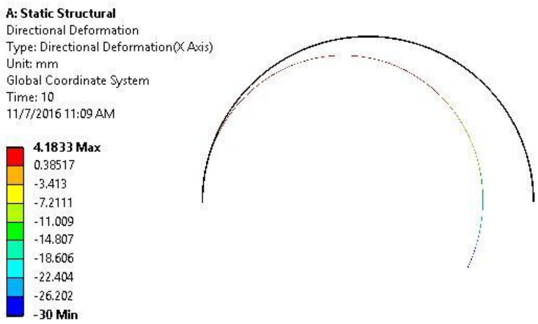

Fig. 10 The deformation of the curved beam at A_3.

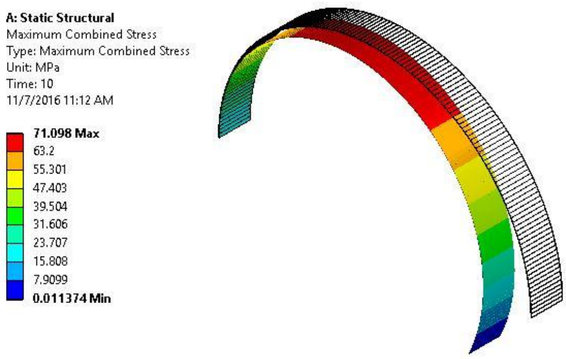

Fig. 11 The stress of the curved beam at A_3.

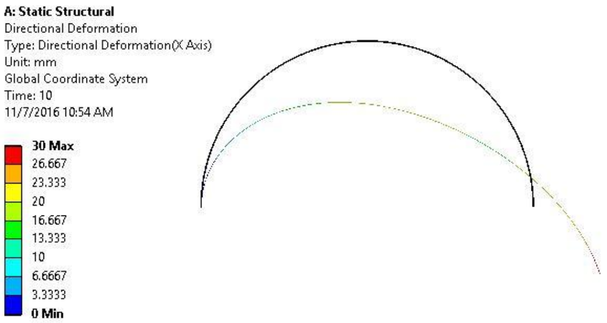

Fig. 12 The deformation of the curved beam at A_4.

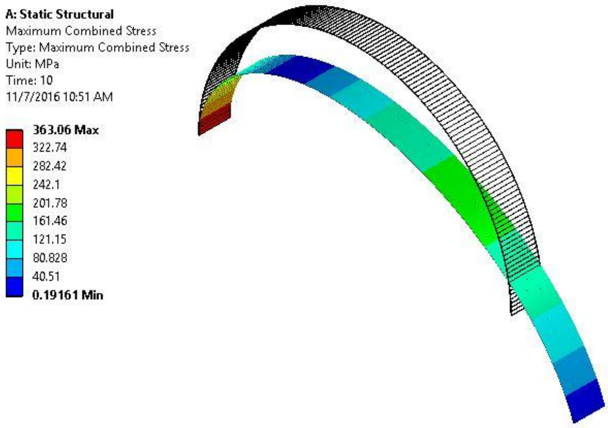

Fig. 13 The stress of the curved beam at A_4.

IV. SHAPE DESIGN OF SLENDER CURVED BEAMS

As shown in Figure 13, the maximum stress within the deformed slender curved beam at A_4 is above the yield strength of the beam material. This maximum stress can be reduced by either increasing the arc length or decreasing the cross section depth of the slender curved beam. Both ways belong to shape design of a slender curved beam that is to improve its performance and better meet its needs and requirements by changing its geometric parameters.

The central angle of the slender circular beam analyzed last section is 180°. The angle is increased to 270° in this section while other parameters of the beam remain unchanged. Figure 14 shows the slender curved beam with the increased arc length. The meshing model of the increased curved beam is shown in Figure 15.

When A of the increased curved beam is displaced to A_1, the deformed shape of the beam is shown in Figure 16. The maximum stress in the deformed beam is 76.54 MPa that is shown in Figure 17, which is smaller than that of 114.61 MPa for the half a circle case shown in Figure 7. To generate the displacements at A, the required input forces (F_x and F_y) are 0.01223 N and 0.10798 N, respectively.

When A of the increased curved beam is displaced to A_2, the deformed shape of the beam is shown in Figure 18. The maximum stress in the deformed beam is 65.154 MPa that is shown in Figure 19, which is much lower than that of 246.69 MPa for the half a circle case in Figure 9. The input forces (F_x and F_y) are -0.01156 N and 0.09961 N, respectively.

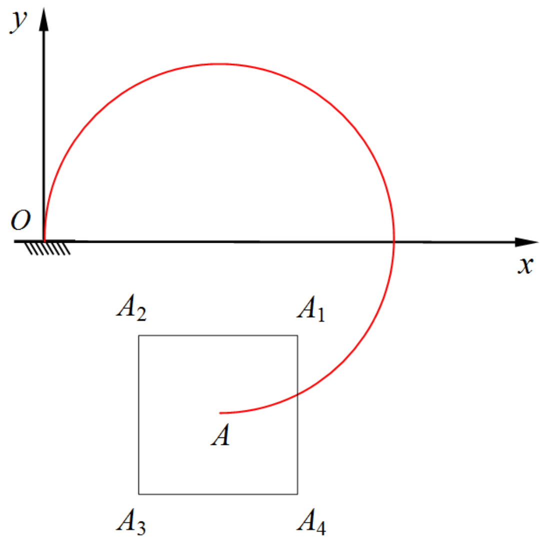

Fig. 14 The slender curved beam with increased length.

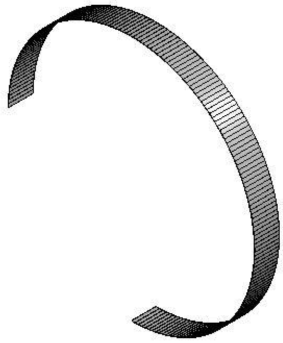

Fig. 15 The meshing model of the increased curved beam.

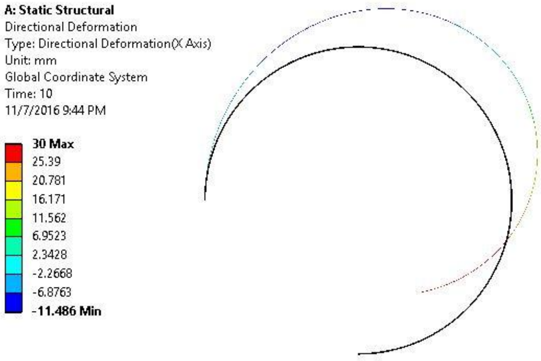

Fig. 16 The deformation of the increased curved beam at A_1.

When A of the increased curved beam is displaced to A_3, the deformed shape of the beam is shown in Figure 20. The maximum stress in the deformed beam is 86.245 MPa that is shown in Figure 21, which is above that of 71.098 MPa for the half a circle case in Figure 10. The input forces (F_x and F_y) are -0.06167 N and -0.1382 N, respectively.

International Journal of Engineering Research & Technology (IJERT)
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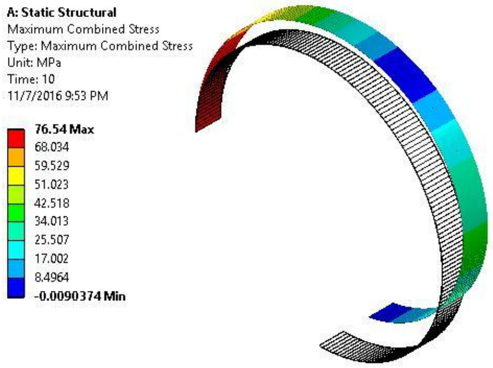

Fig. 17 The stress of the increased curved beam at A_1.

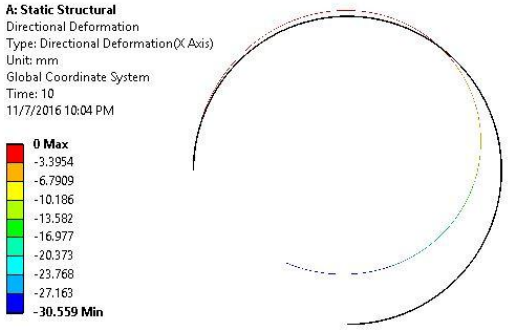

Fig. 18 The deformation of the increased curved beam at A_2.

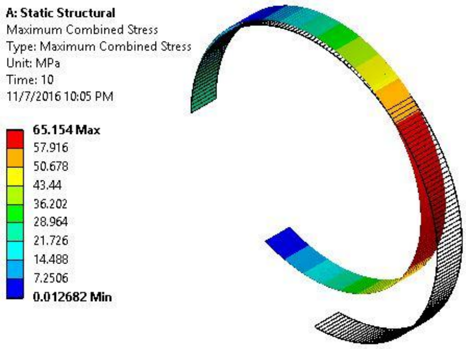

Fig. 19 The stress of the increased curved beam at A_2.

When A of the increased curved beam is displaced to A_4, the deformed shape of the beam is shown in Figure 22. The maximum stress in the deformed beam is 68.005 MPa that is shown in Figure 23, which is far below that of 363.06 MPa for the half a circle case in Figure 13. The input forces (F_x and F_y) are 0.039885 N and -0.12613 N, respectively.

Among all four displacements, the maximum stress within the deformed curved beam with increased arc length is 86.245 MPa, which is well below the yield strength of 350 MPa for the beam material.

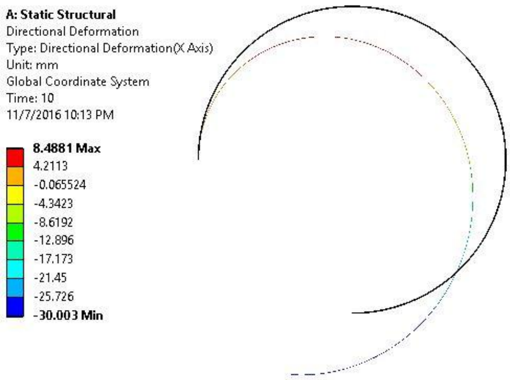

Fig. 20 The deformation of the increased curved beam at A_3.

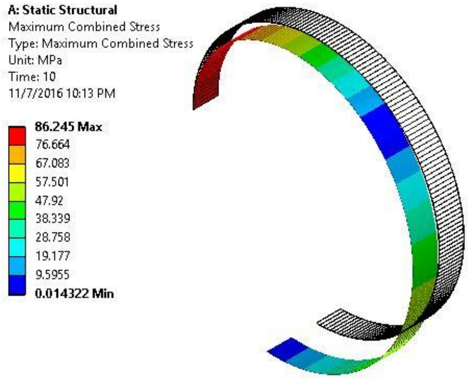

Fig. 21 The stress of the increased curved beam at A_3.

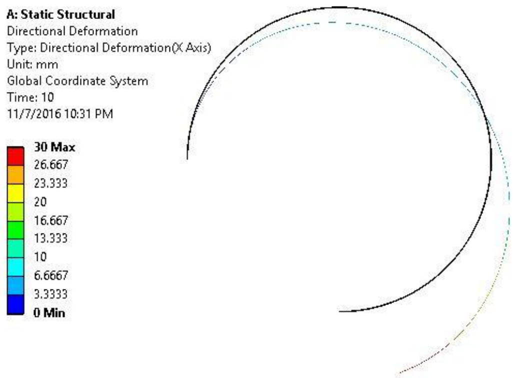

Fig. 22 The deformation of the increased curved beam at A_4.

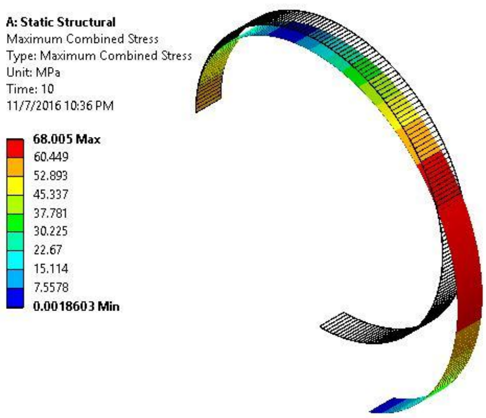

Fig. 23 The stress of the increased curved beam at A_4.

V. CONCLUSIONS

When slender curved beams have much smaller depth of cross section than radius of curvature, their neutral and centroidal axes can be reasonably considered to coincide for stress and deformation analyses. Slender curved beams usually undergo large deformations and have nonlinear load-deflection relationships. Because of the large deformation, the bending moment on a cross section has to be calculated from the deformed shape in order to have a decent accuracy. Large deformation might cause high bending stress within a deformed slender curved beam. The high bending stress can be significantly reduced by either increasing the beam length or decreasing the cross section depth. The stress and deformation analyses presented in the paper provide guidelines for analysing and designing slender curved beams.

ACKNOWLEDGMENT

The authors of this paper gratefully acknowledge the research instrument support of the US National Science Foundation under Grant No. 1337620. Any opinions, findings, recommendations or conclusions expressed in this paper are those of the authors and do not necessarily reflect the views of the US National Science Foundation.

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