Analysis of Cross M- QAM OFDM systems under Gaussian and Impulsive Noise

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***Abstract*—Power line communications (PLC) allows establishing a digital communications without adding any new wires. However, the power line channel was not originally meant to be used for communications and it is a difficult channel for several reasons, for example, it has high noise. This paper derives the exact symbol error probability (SEP) for cross quadrature amplitude modulation with orthogonal frequency division multiplexing (M-QAM/OFDM) systems over Gaussian and impulsive noise. Analytical performances are evaluated and compared with computational simulations, which show good agreement.**

Keywords—M-QAM, OFDM, Gaussian noise, impulsive noise, SEP.

I. INTRODUCTION

The power line communication (PLC) provides a high data transmission rate using quadrature amplitude modulation (QAM) presenting high bandwidth efficiency. If there is a requirement for the transmission of an odd number of bits per symbol, the rectangular QAM is not a good choice in terms of power efficiency. The cross-QAM constellation has been found to be useful in adaptive modulation schemes wherein the constellation size is adjusted depending on the channel quality. For the latter, as the channel quality improves, the constellation size should be from 2^b to 2^{b+2} if only the square QAM were used. By using cross QAM, however, the increments can be changed from 2^b to 2^{b+1}, which allows for more granularities [1]. Cross-QAM is of current interest for employment in adaptive orthogonal frequency division multiplexing (OFDM) systems.

Various research results available in the technical literature evidence that the properties of the noise affecting indoor power line channels. OFDM can perform better than single carrier when the channel is interfered by impulsive noise, because it spreads the effect of impulsive noise over multiple symbols due to discrete Fourier transform (DFT) algorithm.

Therefore, in this paper, we present exact expression for the symbol error probability (SEP) of cross M-QAM and OFDM systems for the channel with Gaussian and impulsive noise.

The remaining part of this paper is organized as follows. In Section II, the combined noise model is presented. The formulas for the SEP of cross M-QAM are derived in section III. Section IV presents a generalized performance expression form OFDM system, taking the effects of the Gaussian and impulsive noise. In Section V, numerical results of the SEP performance of the cross M-QAM/OFDM system are presented. Finally, Section VI gives conclusion.

II. COMBINED NOISE MODEL

Both background noise and impulsive noise are considered when studying the effects of impulsive noise on PLC as their effects differ from that on the OFDM system.

A. Gaussian Noise

Assuming background noise is considered to be additive white Gaussian noise with potential spectral density (PSD) can be expressed as [2]

$$S_w(f) = \sigma_w^2 = \frac{N_0}{2} = \frac{kT_e}{2}[W/Hz] \quad (1)$$

where,

k - the *Boltzman´s* constant ($1{,}38.10^{23}$ [J/k]),

T_e - the system temperature [k],

m_w - mean zero

$2\,\sigma_w^2$ - variance.

The noise is Gaussian with a probability density function (PDF) defined as follows

$$G(x, m_w, \sigma_w^2) = \frac{1}{\sigma_w\sqrt{2\pi}}\exp\left[\frac{-(x - m_w)^2}{2\sigma_w^2}\right] \quad (2)$$

B. Impulsive Noise

The impulsive noises have a behavior in time similar to a sinusoidal wave, with peak values declining exponentially. The occurrence is a Bernoulli process while the amplitudes are modeled as a Gaussian process as follows:

$$i_k = B_k.g_k \quad (3)$$

where, B_k is the Bernoulli process, sequence of zeros and ones with $P_r(B_k = 1) = p$ and g_k is complex white Gaussian noise with mean zero, variance $2\sigma_i^2$ and the PSD is given by

$$S_i(f) = \sigma_i^2 = \frac{N_i}{2} \tag{4}$$

The presence of impulsive noise may lead to severe deterioration in communication performance at PLC receivers.

C. Model

All of the above random sequences are assumed to be independent of each other. This model can be physically thought of as each transmitted data symbol being hit independently by an impulsive with probability p and with random amplitude. Hence, the combined noise seen by the receiver is [3].

$$n_k = n_{kR} + jn_{kI} = w_k + i_k \tag{5}$$

where,

n_k - noise with real and imaginary part,

w_k - *Gaussiano* noise,

i_k - impulsive noise.

The ratio between the noises is

$$\mu = \frac{N_i}{N_0} \tag{6}$$

and the total noise is a random variable with variance described as follows

$$\sigma_m{}^2 = \sigma_w^2 + \sigma_i^2 = \frac{N_0}{2} + \frac{N_i}{2} = \frac{N_0}{2}(1+\mu) \tag{7}$$

III. SEP OF CROSS-M-QAM

QAM is extensively used modulation scheme in which both amplitude and phase will carry information. The constellation has $M = 2^b$ symbols, where the number of bits in each constellation symbol is b [4]. In the cross M-QAM constellation, b is an odd integer. The alphabet was created in the range m

$$\gamma_{M-QAM} = \{\mp(2m-1) \mp j(2m-1)\} \quad m \in \left\{1, \ldots, \frac{3}{4}\sqrt{\frac{M}{2}}\right\} \tag{8}$$

±11±j11	±11±j9	±11±j7	±11±j5	±11±j3	±11±j1
±9±j11	±9±j9	±9±j7	±9±j5	±9±j3	±9±j1
±7±j11	±7±j9	±7±j7	±7±j5	±7±j3	±7±j1
±5±j11	±5±j9	±5±j7	±5±j5	±5±j3	±5±j1
±3±j11	±3±j9	±3±j7	±3±j5	±3±j3	±3±j1
±1±j11	±1±j9	±1±j7	±1±j5	±1±j3	±1±j1

The energy of the individual alphabets is

$$E_1 = \sum_{m=1}^{m=\frac{3}{4}\sqrt{\frac{M}{2}}} |(2m-1)+j(2m-1)|^2 = 2\sum_{m=1}^{m=\frac{3}{4}\sqrt{\frac{M}{2}}} |(2m-1)|^2 \tag{9}$$

$$= \frac{2}{3}s(4s^2-1) = \frac{1}{2}\left(\sqrt{\frac{M}{2}}\right)\left(\frac{9M}{8}-1\right)$$

each alphabet is used $\left\{3\sqrt{\frac{M}{2}}\right\}$ times in the cross M-QAM constellation and total energy { E_{1T} } is given by

$$E_{1T} = \left[\frac{1}{2}\left(\sqrt{\frac{M}{2}}\right)\left(\frac{9M}{8}-1\right)\right].3\sqrt{\frac{M}{2}} = \frac{3M}{4}\left(\frac{9M}{8}-1\right) \tag{10}$$

The 128-QAM constellation is shown in the Fig. 1.

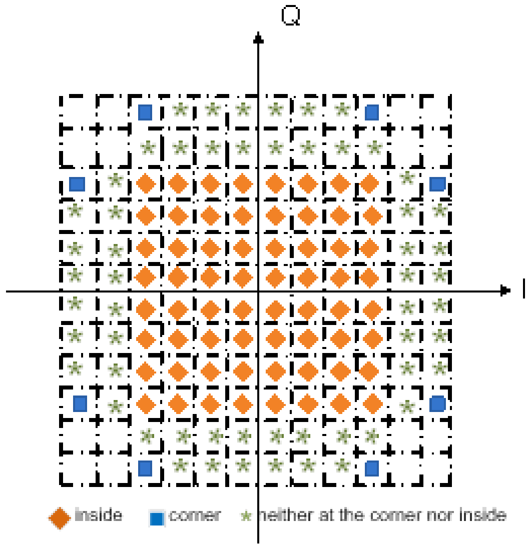

Fig. 1 Cross 128-QAM constellation modified [4].

There are non-constellation individual alphabets {±11±j11, ±11±j9, ±9±j11, ±9±j9} and the energy is

$$E_2 = 2\sum_{m=\frac{1}{2}\sqrt{\frac{M}{2}}+1}^{m=\frac{3}{4}\sqrt{\frac{M}{2}}} (2m-1)^2 = \frac{13}{4}M + 4 - 2\sqrt{\frac{M}{2}} \tag{11}$$

and each alphabet is used $\left\{\sqrt{\frac{M}{2}}\right\}$ times in the cross M-QAM constellation and total energy { E_{2T} } is given by

$$E_{2T} = \left[\frac{13}{4}M + 4 - 2\sqrt{\frac{M}{2}}\right].\sqrt{\frac{M}{2}} = \tag{12}$$

$$= \frac{M}{128}\left[(26M+32) - M\sqrt{\frac{128}{M}}\right]$$

The average energy from M constellation symbols { $E_{M-QAM(Cross)}$ } is the sum of (10) and (12) divided by M. The average energy can be evaluated by

$$E_{M-QAM(Cross)} = \frac{E_{1T} - E_{2T}}{M} = \frac{1}{128}\left(82M + M\sqrt{\frac{128}{M}} - 128\right) \quad (13)$$

The average energy and scaling factor {k} are tabulated in Table I.

TABLE I. SCALING FACTOR.

M-QAM	$E_{M\text{-}QAM}$(Cross)	Scaling factor (k)
32-QAM	20	1/√20
128-QAM	82	1/√82
512-QAM	330	1/√330

There are three types of constellation symbols in a non-square M-QAM constellations as depicted in Fig. 1.

Constellation points in the corner: has two neighboring symbols. The number of constellation points in the corner in any cross M-QAM constellation is always 8.

$$N_2 = 8 \quad (14)$$

The symbol in the corner is decoded correctly with the region's integration within the limits, Fig. 2.

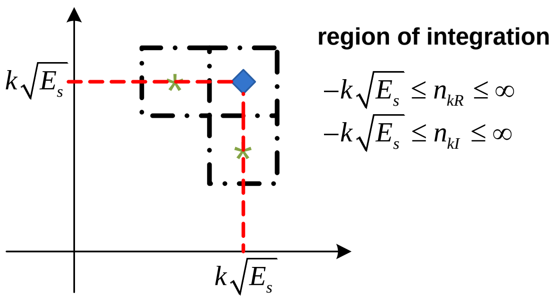

Fig. 2 - The limits of integration to the symbols with two neighbors.

The limits of integration are applied in the probability distribution function

$$P_e = 1 - \int_{-k\sqrt{E_s}}^{+k\sqrt{E_s}} \int_{-k\sqrt{E_s}}^{+k\sqrt{E_s}} p_n(n_R, n_I) dn_R dn_I \quad (15)$$

and the probability of error of symbol in the corner is given by

$$p_e(N_2) = erfc\left(k\sqrt{\frac{E_s}{N_0}}\right) - \frac{1}{4} erfc^2\left(k\sqrt{\frac{E_s}{N_0}}\right) \quad (16)$$

Constellation points in the inside: has four neighboring symbols. The number of constellation points in the insider in any cross M-QAM constellation is

$$N_4 = M - 3\sqrt{2M} + 8 \quad (17)$$

The symbol in the inside is decoded correctly with the region's integration within the limits, Fig. 3.

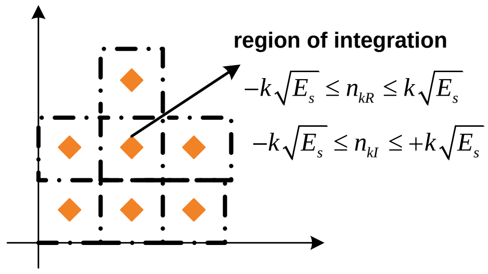

Fig. 3 - The limits of integration to the symbols with four neighbors.

The limits of integration are applied in (15) and the probability of error of symbol in the inside is given by

$$p_e(N_4) = 2erfc\left(k\sqrt{\frac{E_s}{N_0}}\right) - erfc^2\left(k\sqrt{\frac{E_s}{N_0}}\right) \quad (18)$$

Constellation points neither at the corner, nor at the center: has three neighboring symbols. The number of constellation points in the neither at the corner, nor at the center in any cross M-QAM constellation is

$$N_3 = 3\sqrt{2M} - 16 \quad (19)$$

The symbol in the neither at the corner, nor at the center is decoded correctly with the region's integration within the limits, Fig. 4.

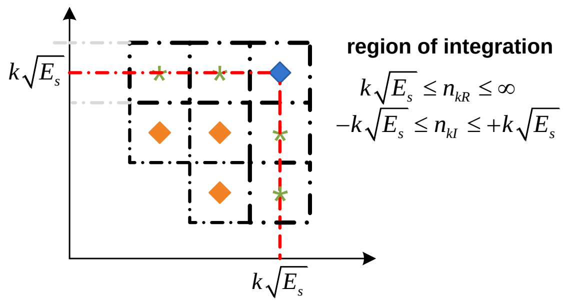

Fig. 4 - The limits of integration to the symbols with three neighbors.

The probability of error of symbol in the neither at the corner, nor at the center is defined as

$$p_e(N_3) = \frac{3}{2} erfc\left(k\sqrt{\frac{E_s}{N_0}}\right) - \frac{1}{2} erfc^2\left(k\sqrt{\frac{E_s}{N_0}}\right) \quad (20)$$

The Symbol Error Probability (SEP) of cross M-QAM follow the logic:

$$P_{e(M-QAMcruz)} = \frac{eq.\{[(14)\times(16)] + [(17)\times(18)] + [(19)\times(20)]\}}{M}$$

combining results leads the exact SEP due to additive white Gaussian noise (AWGN).

$$P_{e(M-QAMcruz)} = \left(2-\frac{3}{2}\sqrt{\frac{2}{M}}\right)erfc\left(k\sqrt{\frac{E_s}{N_0}}\right) - \left(1+\frac{2}{M}-\frac{3}{2}\sqrt{\frac{2}{M}}\right)erfc^2\left(k\sqrt{\frac{E_s}{N_0}}\right) \quad (21)$$

The proposed equation will be validated using other published equations. An exact expression to determine the SEP of 32-QAM cross on a channel with Gaussian noise is proposed in [5].

$$P_e = \frac{4}{32}Q\left(\sqrt{\frac{E_A}{10\sigma_w^2}}\right)+\frac{104}{32}Q\left(\sqrt{\frac{E_A}{20\sigma_w^2}}\right) - \frac{92}{32}Q^2\left(\sqrt{\frac{E_A}{20\sigma_w^2}}\right) \quad (22)$$

where,

$E_A = 20d^2$ - average energy,

d - minimum distance between two adjacent points in the constellation.

The equations (23) e (24) is an exact intuitive geometric infinite double series for the average symbol error probability of 128-cross-QAM and 512-cross-QAM in additive white Gaussian noise is derived in [6].

$$P_e \approx \frac{1}{128}\left[468Q\left(\sqrt{\frac{\gamma_s}{41}}\right)-440Q^2\left(\sqrt{\frac{\gamma_s}{41}}\right)\right] \quad (23)$$

$$P_e \approx \frac{1}{512}\left[1956Q\left(\sqrt{\frac{\gamma_s}{165}}\right)-1880Q^2\left(\sqrt{\frac{\gamma_s}{165}}\right)\right] \quad (24)$$

where,

$$\gamma_s = \frac{E_s}{N_0} = \frac{82d^2}{2\sigma_w^2} \quad (25)$$

Applying (6) and (7) into (21) leads to the symbol error probability of cross M-QAM due to AWGN and impulsive noise.

$$P_{e(M-QAMcross)} = \left(2-\frac{3}{2}\sqrt{\frac{2}{M}}\right)erfc\left(\frac{k}{\sqrt{1+\mu}}\sqrt{\frac{E_s}{N_0}}\right) - \left(1+\frac{2}{M}-\frac{3}{2}\sqrt{\frac{2}{M}}\right)erfc^2\left(\frac{k}{\sqrt{1+\mu}}\sqrt{\frac{E_s}{N_0}}\right) \quad (26)$$

IV. SEP OF CROSS M-QAM

In OFDM multiple sinusoidals with frequency separation $\frac{1}{T}$ is used and can be defined as [4]

$$s_k = \frac{1}{\sqrt{N_c}}\sum_{n=0}^{N_c-1} a_n \exp\left(\frac{j2\pi nk}{N_c}\right) \quad (27)$$
$$k = 0,1,\ldots,N_c-1$$

where,

s_k - transmitted signal,

a_n - n^{th} symbol in the message,

N_c - number of carriers.

The signal received $\{r_k\}$ with influence of noise is solved as follows

$$r_k = \frac{1}{\sqrt{N_c}}\sum_{n=0}^{N_c-1} a_n \exp\left(\frac{j2\pi nk}{N_c}\right)+w+i \quad (28)$$
$$k = 0,1,\ldots,N_c-1$$

where,

w - Gaussian noise,

i - impulsive noise.

The characteristic function is formulated as

$$\Phi_{Nk}(\omega_1,\omega_2) = \sum_{m=0}^{N_c}\binom{N_c}{m}p^m(1-p)^{N_c-m}\exp\left[\frac{-\sigma_m^2\left(\omega_1^2+\omega_2^2\right)}{2}\right] \quad (29)$$

where, the variance of the combined noise is developed considering the noise and the number of OFDM subcarriers [5].

$$\sigma_m{}^2 = \sigma_w^2 + \frac{m\sigma_i^2}{N_c} = \frac{N_0}{2}+\frac{mN_i}{2N_c} = \frac{N_0}{2}\left(1+\frac{m\mu}{N_c}\right) \quad (30)$$

Applying (30) into (21) leads to the symbol error probability of cross M-QAM

$$P_e = 1-\left\{\sum_{m=0}^{N}\binom{N_c}{m}p^m(1-p)^{N_c-m} \left[\left(2-\frac{3}{2}\sqrt{\frac{2}{M}}\right)erfc(a)-\left(1-\frac{3}{2}\sqrt{\frac{2}{M}}+\frac{2}{M}\right)erfc^2(a)\right]\right\} \quad (31)$$

V. NUMERICAL RESULTS

In the first stage of simulations, we have verified our exact SEP expressions. The Figs. 5, 6 and 7 presents the results obtained by simulation of cross M-QAM modulations due to AWGN.

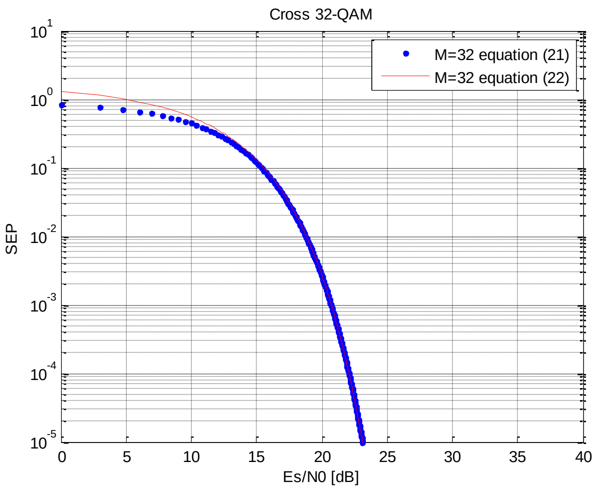

Fig. 5 – 32-cross-QAM.

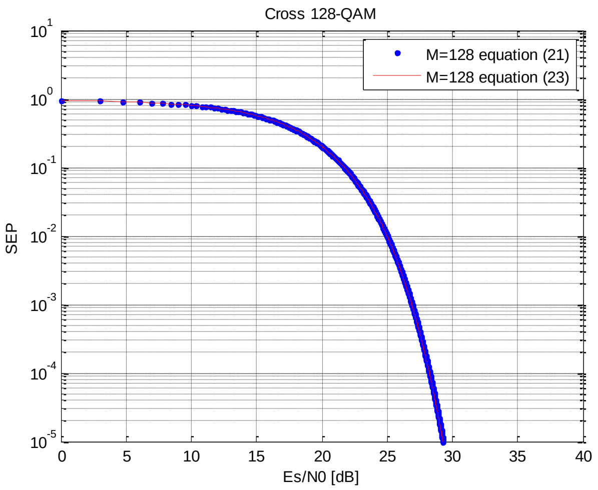

Fig. 6 – 128-cross-QAM.

The figures shows excellent agreement between our results with other published results for cross 32-QAM [5], 128-QAM and 512-QAM in [6].

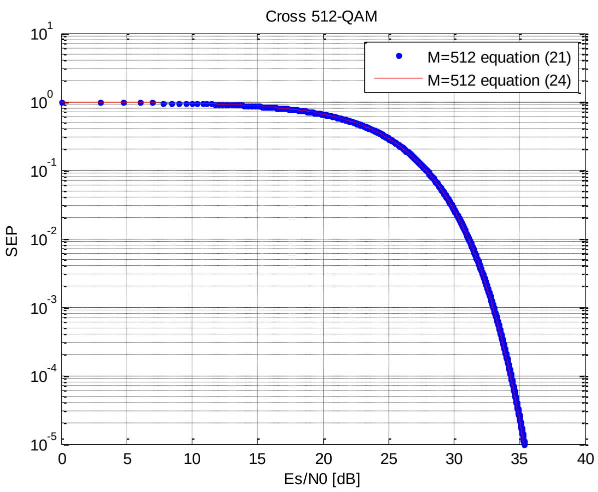

Fig. 7 – 512-cross-QAM.

The analysis was made considering SEP $\{10^{-5}\}$. The ratio $\left\{\frac{E_s}{N_0}\right\}$ was 23,1dB, 29,2dB and 35,3dB in the Figs. 5, 6 and 7, respectively.

The Figs. 8 and 9 presents the results obtained by simulation of cross M-QAM due to Gaussian and impulsive noises. In the article [7], the authors used the ratio between the Gaussian noise and the impulsive with the value of $\{\sigma_w^2 = 10^{-2}\sigma_i^2\}$. The results presented in the Fig. 8.

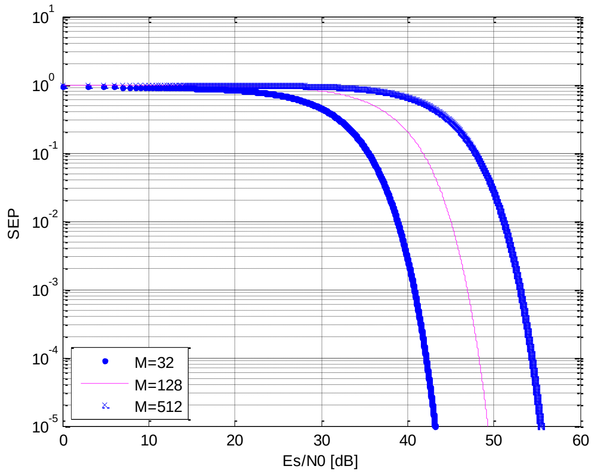

Fig. 8 – SEP performances of 32/128/512-QAM with μ=100.

In [3] specify the ratio between the variance of Gaussian noise and impulse $\{\sigma_i^2 = 10\sigma_w^2\}$. From Fig. 9, it can be seen that the addition of the impulsive noise increased the ratio $\left\{\frac{E_s}{N_0}\right\}$. The new values are about 33,5dB, 39,6dB and 45,7dB.

To improve the ratio, we add OFDM with 52 subcarriers. The results obtained by simulation of cross M-QAM/OFDM due to Gaussian and impulsive noises are presents in the Figs. 10, 11 and 12. The impulsive noise occurrence probability ranged from {0.01; 0.1; 1.0}.

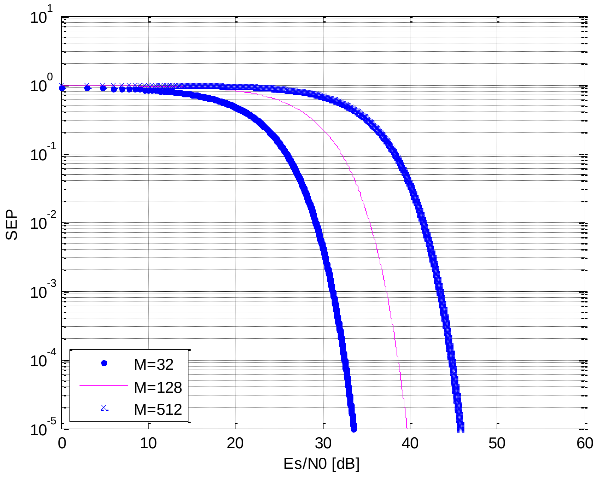

Fig. 9 – SEP performances of 32/128/512-QAM with μ=10.

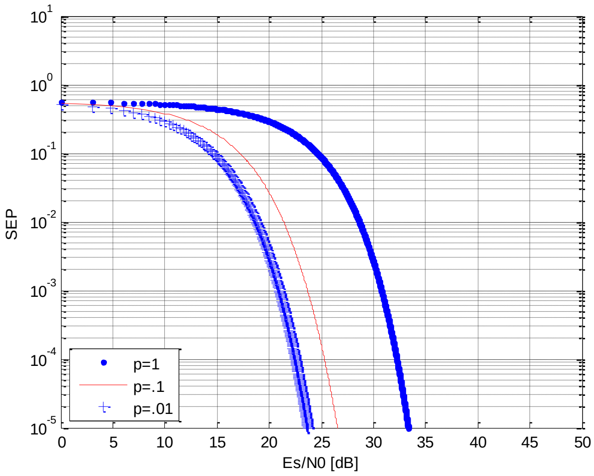

Fig. 10 – SEP performances of 32-QAM/OFDM.

From Fig. 10, it can be seen that the new value is about 24 to 34dB. Fig. 11 shows a variation of 30dB to 40dB.

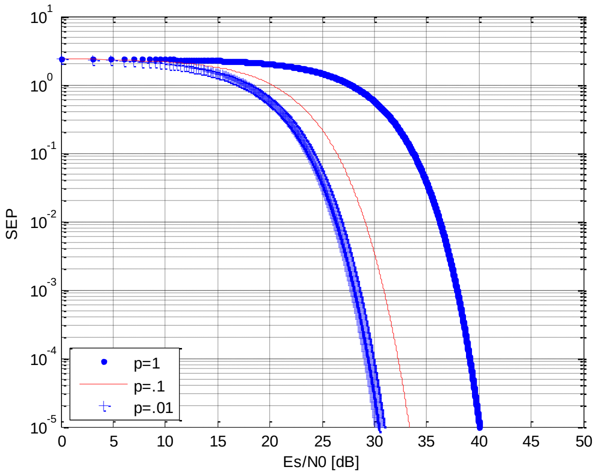

Fig. 11 – SEP performances of 128-QAM/OFDM.

As has been verified in Fig. 12, the ratio $\left\{\frac{E_s}{N_0}\right\}$ was between 35dB and 46dB. Hence, OFDM can achieve good performance under the impulsive noise effect.

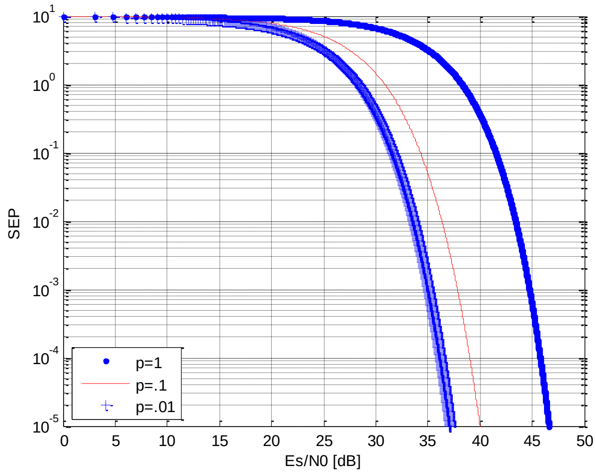

Fig. 12 – SEP performances of 512-QAM/OFDM.

VI. CONCLUSION

In this paper, we have developed form expression for the SEP of cross M-QAM constellations in the presence of Gaussian noise. We have compared the exact SEP expression with other expressions published literature. Then we evaluate the performance of cross M-QAM in the presence of impulsive and AWGN noise. The addition of impulse noise increased a SEP of cross M-QAM. To control the SEP, we use OFDM.

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