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Analysis of Fractional Order SIR Model

A. George Maria Selvam [1], Britto Jacob. S[3]
[1, 3] Sacred Heart College,
Tirupattur-635 601, S. India.

D. Abraham Vianny[2]
[2] Knowledge Institute of Technology,
Kakapalayam – 637 504, S. India

***Abstract*— Fractional order SIR epidemic model is considered for dynamical analysis. The basic reproductive number is established and an analysis is carried out to study the stability of the equilibrium points. The time plots and phase portraits are provided for different sets of parameter values. Numerical simulations are presented to illustrate the stability analysis using Generalized Euler Method.**

Keywords—Fractional order, SIR model , Differential equations, Stability, Generalized Euler method.

1. FRACTIONAL DERIVATIVES AND INTEGRALS

Fractional Calculus is a branch of mathematics that deals with the study of integrals and derivatives of non-integer orders, plays an outstanding role and have found several applications in large areas of research during the last decade. Behavior of many dynamical systems can be described and studied using the fractional order system. Fractional derivatives describe effects of memory. This section presents some important definitions of fractional calculus which arise as natural generalization of results from calculus.

Definition 1. The Riemann – Liouville fractional Integral of order $0 \le \alpha \le 1$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{(\alpha-1)} f(u)\, du, t > 0$$

Definition 2. The Riemann – Liouville fractional derivative is defined as

$$D_t^{\alpha} f(t) = \frac{d}{dt} J^{1-\alpha} f(t)$$

Definition 3. The Caputo fractional derivative is defined as

$$D_t^{\alpha} f(t) = J^{1-\alpha} \frac{d}{dt} f(t)$$

If m and n are integers such that $m > n$, then n^{th} order derivative of t^m (using Euler's Gamma Function) is

$$\frac{d^n}{dt^n} t^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}.$$

2. GENERALIZED TAYLORS FORMULA AND EULER METHOD

We introduce a generalization of Taylor's formula that involves Caputo fractional derivatives. Suppose that $D^{k\alpha} f(x) \in C(0,a]$, for $k = 0,1,\dots,n+1$, where $0 < \alpha \le 1$.

Then we have

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{(D^{(n+1)\alpha} f)(\zeta)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha}, 0 \le \zeta \le x, \forall x \in C(0,a] \quad (1)$$

In case of $\alpha = 1$, the generalized Taylor's formula (1) reduces to the classical Taylor's formula.

Zaid and Momani derived the generalized Euler's Method for the numerical solution of initial value problems with Caputo derivatives. The method is a generalization of the classical Euler's method. Consider the following general form of IVP;

$$D^{\alpha} y(t) = f(t, y(t)), \qquad y(0) = y_0 \qquad (2)$$

for $0 < \alpha \le 1, 0 < t < a$. The general formula for Generalized Euler's Method (GEM) is

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\left(t_j, y(t_j)\right), j = 0,1,\dots,n-1 \quad (3)$$

If $\alpha = 1$ then the generalized Euler's method (3) reduces to the classical Euler's method.

3. MODEL DESCRIPTION

Mathematical modeling is used to analyze, study the spread of infectious diseases and predict the outbreak and to formulate policies to control an epidemic. We obtain fractional SIR epidemic model by introducing fractional derivative of order $\alpha (0 < \alpha \le 1)$ in the classical SIR epidemic equations. In this paper, we study fractional order SIR epidemic model with vaccination and treatment. The total population N is partitioned into three compartments which are Susceptible, Infected and Recovered with sizes denoted by N $S(t), I(t)$ and $R(t)$[4].

Variable	Meaning
$S(t)$	The number of Susceptible Individuals at time t
$I(t)$	The number of Infectious Individuals at time t
$R(t)$	The number of Recovered Individuals at time t
Parameters	Meaning
μ	Birth rate or Recruitment rate
δ	Infectious period
β	Contact rate

By introducing fractional order, $D^{\alpha}S, D^{\alpha}I$ and $D^{\alpha}R$ are the derivatives of $S(t), I(t)$ and $R(t)$ respectively, of arbitrary order α in sense of Caputa and $0 < \alpha < 1$, then the system leads to fractional equations given by,

$$D^{\alpha}S(t) = \mu - \beta S(t)I(t) - \mu S(t)$$
$$D^{\alpha}I(t) = \beta S(t)I(t) - \delta I(t) - \mu I(t) \qquad (4)$$
$$D^{\alpha}R(t) = \delta I(t) - \mu R(t)$$

Where $S(0) = S_0, I(0) = I_0$ and $R(0) = R_0$.

The system of fractional differential equations above is reduced to an integer order system by setting $\alpha = 1$. Also we have

$$D^{\alpha}N(t) = \mu - \mu N(t)$$

Since $N(t) = S(t) + I(t) + R(t)$, $R(t)$ can always be obtained by the equation

$$R(t) = N(t) - S(t) - I(t)$$

We now consider the system of equations

$$D^{\alpha}S = \mu - \beta SI - \mu S$$
$$D^{\alpha}I = \beta SI - \delta I - \mu I \qquad (5)$$
$$D^{\alpha}N = \mu - \mu N$$

with $S(0) = S_0, I(0) = I_0, N(0) = N_0$, where $0 < \alpha < 1$.

4. BASIC REPRODUCTIVE NUMBER

The basic reproduction number is the average number of Secondary cases generated by a typical infective within a population with no immunity to the disease. It is denoted by R_0 [1, 2, 5]. If $R_0 < 1$, then the disease dies out. If $R_0 > 1$, the it implies that the disease spreads in the susceptible population.

We determine (R_0) by using the Next Generation Matrix (NGM) Approach. The Next Generation Matrix is given by $K = FV^{-1}$, $F = \begin{bmatrix} \beta S & 0 \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} \delta + \mu & -\beta I \\ \beta S & \beta I + \mu \end{bmatrix}$

$$K = FV^{-1} = \begin{bmatrix} \dfrac{(\beta S_0)(\beta I_0 + \mu)}{(\delta + \mu)(\beta I_0 + \mu) + (\beta S_0)(\beta I_0)} & 0 \\ 0 & 0 \end{bmatrix}$$

(6)

At Disease Free Equilibrium (DFE), $S_0 = 1$ and $I_0 = 0$ substitute into the Next Generation Matrix, we get

$$K = FV^{-1} = \begin{bmatrix} \dfrac{\beta}{(\delta + \mu)} & 0 \\ 0 & 0 \end{bmatrix}$$

Since R_0 is the most dominant eigenvalue of Next Generation Matrix, then

$$R_0 = \frac{\beta}{(\delta + \mu)}$$

The DFE is locally stable if $R_0 < 1$, whereas it is unstable if $R_0 > 1$.

5. NON-NEGATIVE SOLUTIONS AND EQUILIBRIUM POINTS

Given the equation of the population as, $D^{\alpha}N = \mu - \mu N$, and from the dynamics described by system of equations (5), the region $R_+^3 = (x \in R^3 : x \geq 0)$ and $x(t) = (S(t), I(t), R(t))^T$ is positively invariant (non-negative solutions).

To evaluate the equilibrium points, we consider

$D^{\alpha}S = 0, D^{\alpha}I = 0, D^{\alpha}N = 0.$

The fractional order system (5) has two equilibria $E_0 = (1,0,1)$ and $E_1 = \left(\frac{1}{R_0}, \frac{\mu}{\beta}(R_0 - 1), 1\right)$.

6. STABILITY OF EQUILIBRIA

Based on (5), the Jacobian matrix of the system is

$$J(S, I, N) = \begin{bmatrix} -\beta I - \mu & -\beta s & 0 \\ \beta I & \beta S - \delta - \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix} \qquad (7)$$

At the Disease-Free Equilibrium (DFE), the Jacobian matrix (7) for E_0 is

$$J(E_0) = \begin{bmatrix} -\mu & -\beta & 0 \\ 0 & \beta - \delta - \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

The eigen values are $\lambda_1 = \lambda_2 = -\mu$ and $\lambda_3 = \beta - \delta - \mu$. The Disease Free Equilibrium point of the system is asymptotically stable if $R_0 < 1$, $|\lambda_{1,2}| < 1$ and $|\lambda_3| < 1$.

At the Endemic Equilibrium, the Jacobian matrix (7) for E_1 is

$$J(E_1) = \begin{bmatrix} -\mu R_0 & -\beta \dfrac{1}{R_0} & 0 \\ \mu(R_0 - 1) & \beta \dfrac{1}{R_0} - \delta - \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

The eigen values are $\lambda_1 = -\mu$ and

$$\lambda_{2,3} = \frac{-\left(\mu R_0 - \beta \frac{1}{R_0} + \delta + \mu\right)}{2} \pm \frac{1}{2}\sqrt{\left(\mu R_0 - \beta \frac{1}{R_0} + \delta + \mu\right)^2 + 4\mu\left[R_0(\mu + \delta) - \left(\beta \frac{1}{R_0} + \frac{1}{\beta} - \beta\right)\right]}$$

The Endemic Equilibrium point of the system is asymptotically stable if $R_0 > 1$, then $|\lambda_1| < 1$, $|\lambda_{2,3}| < 1$.

7. NUMERICAL SIMULATIONS

Numerical solution of the fractional order system is

$$S(t_{j+1}) = S(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(\mu - \beta S(t_j)I(t_j) - \mu S(t_j)\right)$$

$$I(t_{j+1}) = I(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(\beta S(t_j)I(t_j) - \delta I(t_j) - \mu I(t_j)\right)$$

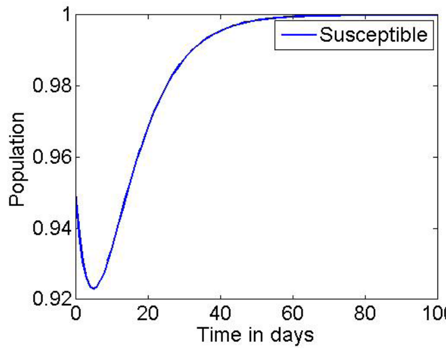
$$N(t_{j+1}) = N(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\left(\mu - \mu N(t_j)\right)$$

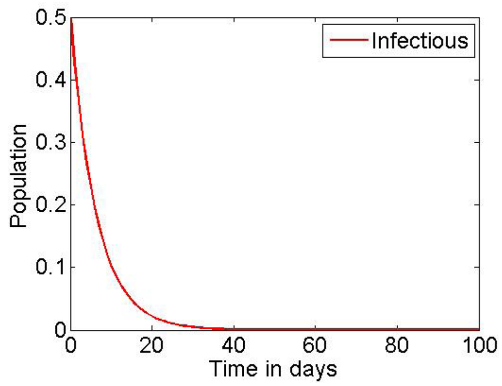
for

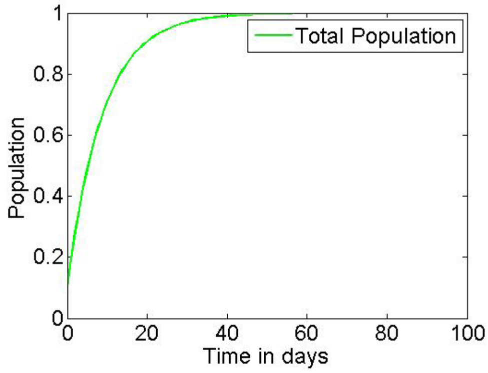
$$j = 0, 1, \ldots, k-1.$$

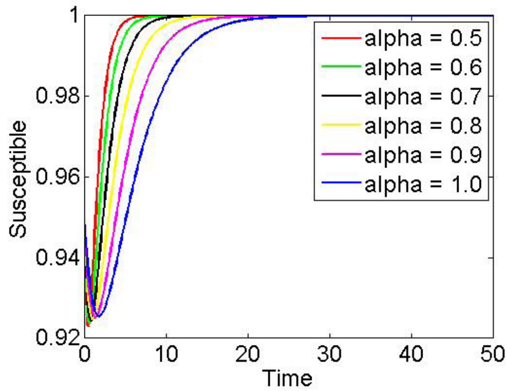
Numerical techniques are used to analyze the qualitative properties of fractional order differential equations since the equations do not have analytic solutions in general.

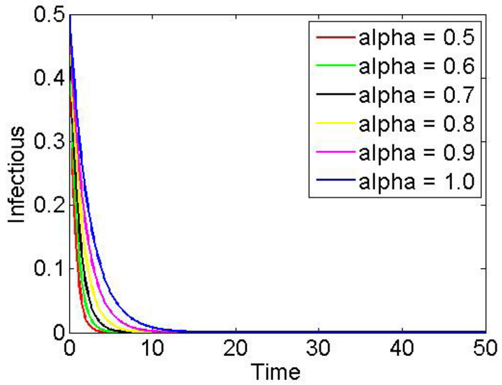
Example 1.Let us consider the parameter values $\mu = 0.3, \beta = 0.3, \delta = 0.2$ *and* $h = 0.1$ *the initial conditions (S(0), I(0), N(0)) = (0.95, 0.05, 1.00) the fractional derivative order* $\alpha = 0.9$. *For these parameter the corresponding eigen values are* $\lambda_{1,2} = -0.3$ *and* $\lambda_3 = -0.2$ *for* E_0. *If* $|\lambda_{1,2}| = 0.3 < 1, |\lambda_3| = 0.2 < 1$ *and* $R_0 = 0.6 < 1$, *then the disease free equilibrium is locally asymptotically stable.*











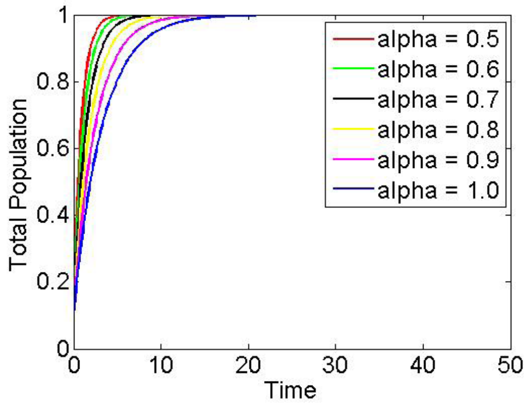
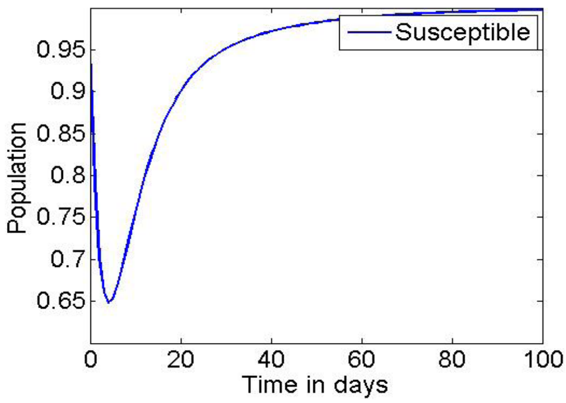
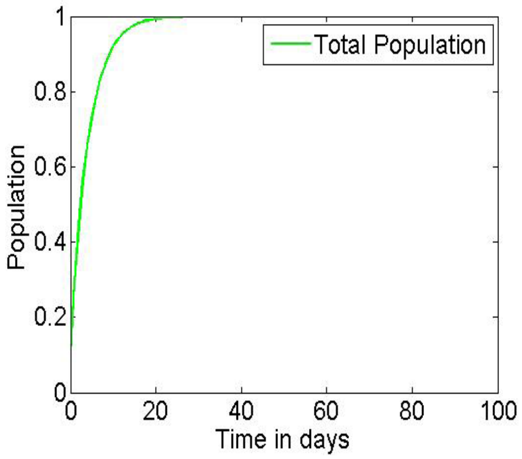


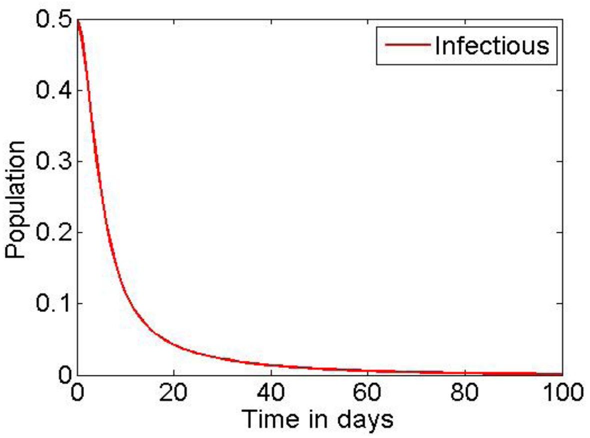
Figure 1.

Example 2. Let us consider the parameter values $\mu = 0.012, \beta = 0.5, \delta = 0.13$ *and* $h = 0.1$ *and the initial conditions (S(0), I(0), N(0)) = (0.95, 0.05, 1.00) the fractional derivative order* $\alpha = 0.9$. *For these parameter the corresponding eigen values are* $\lambda_1 = -0.0120$ *and* $\lambda_{2,3} = -0.0211 \pm 0.0620i$ *for* E_1. *If* $|\lambda_1| = 0.0120 < 1, |\lambda_{2,3}| = 0.0655 < 1$*and*

$R_0 = 3.5211 > 1$, *then the endemic equilibrium is locally asymptotically stable.*







Example 3. Let us consider the parameter values $\mu = 0.026, \beta = 0.7, \delta = 0.33$ *and* $h = 0.1$ *and the initial conditions (S(0), I(0), N(0)) = (0.95, 0.05, 1.00) the fractional derivative order* $\alpha = 0.9$. *For these parameter the corresponding eigen values are* $\lambda_1 = -0.0260$ *and* $\lambda_{2,3} = -0.0256 \pm 0.0911i$ *for* E_1. *If* $|\lambda_1| = 0.0260 < 1, |\lambda_{2,3}| = 0.0946 < 1$ *and* $R_0 = 1.9663 > 1$, *then the endemic equilibrium is locally asymptotically stable.*

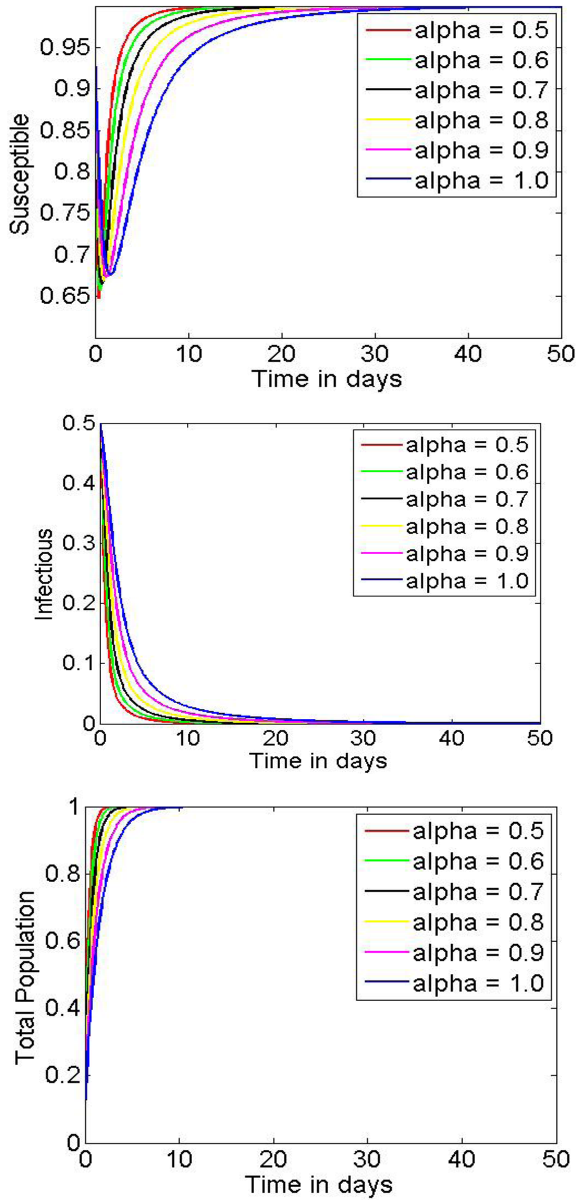
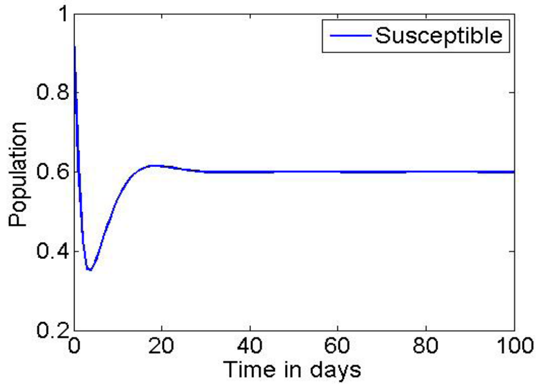
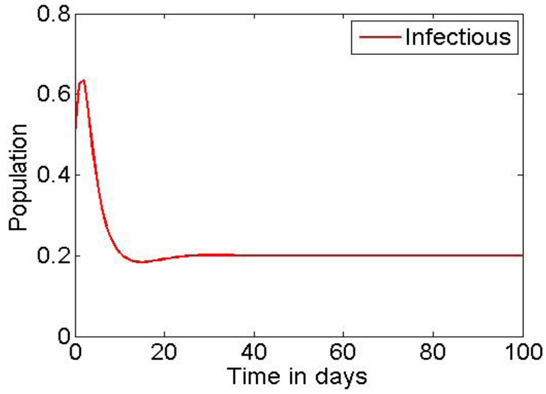
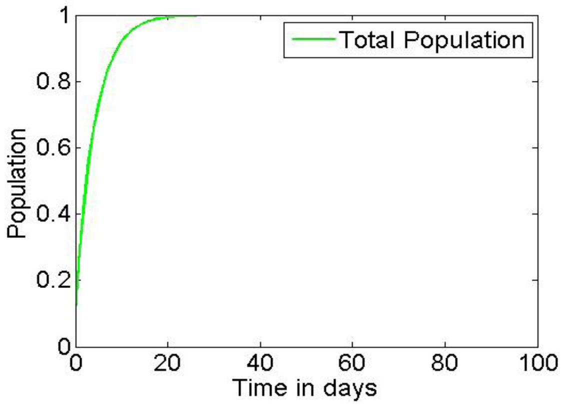
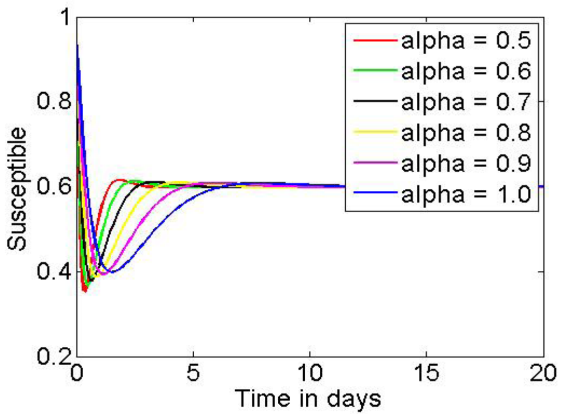


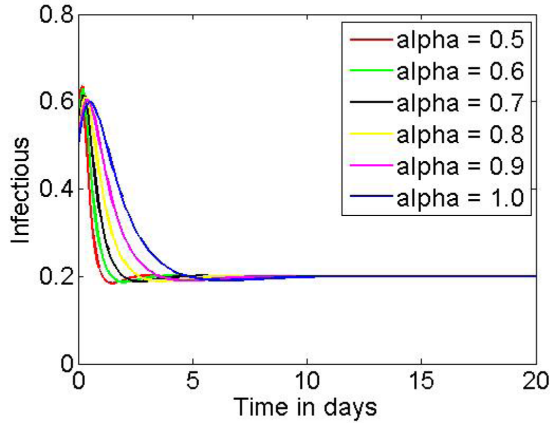
Figure 2.











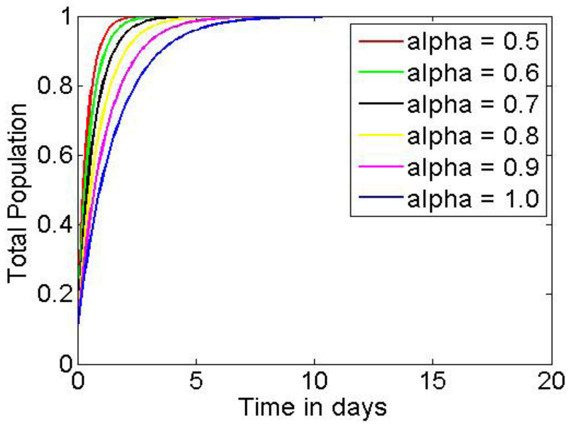


Figure 3.

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