Analysis Of Solving Polynomial Equations Using Homotopy Continuation Method

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ABSTRACT

The Newton's method was used to compute the root of a polynomial equation and then compared with the homotopy continuation method. The imbedding method has the advantages of producing solutions over a large range of the independent variable, thereby giving a complete description of the solution behaviour and is use as a tool in overcoming the local convergence of iterative processes. The imbedding methods may be considered as a possibility to widen the domain of convergence that is providing all the isolated real roots of a polynomial equation.

INTRODUCTION

The necessity of solving nonlinear equation often arises in simulating and designing a chemical plant or optimizing a process. When f(x) = 0 where $f : R^n \to R^n$ is a nonlinear equation, we want to find the solution of f(x). The Newton's method applied to nonlinear equation of the form f(x) = 0 in general will only converge if the iteration is started near a root of the equation. (Kuno and Seader, 1988) said that, Newton's method is locally convergent. In addition, the method is designed to locate, at best, just one root even though multiple solutions may exist. This is the major disadvantage.

 The imbedding method or continuation methods are rapidly being applied to numerous problems of such kind. A solution method for nonlinear equation that is globally convergent from any starting point is the homotopy continuation method. The method has advantages of

producing solution over a large range of independent variables, giving a complete description of the solution behavior and a substantial improvement to Newton's and other similar methods in terms of global convergence. This method is based on the homotopy concept in algebraic topology.

The method, does not solve the nonlinear equation directly, but gradually reaches a root by beginning from a starting point which satisfies the second nonlinear equation. Both equations are embedded or homotoped into a homotopy function. Thus, if f(x) is the nonlinear equation to be solved and g(x) is a second simpler nonlinear equation of same degree with f(x), the homotopy function might be constructed as

$$H(x, t) = tf(x) + (1 - t)g(x) = 0$$

Where t is a scalar homotopy parameter which is gradually varied from 0 to 1 as the path is tracked from starting point to a solution.

Finding roots fails to give a correct solution to a nonlinear algebraic equation unless a good guess is chosen. Difficulty in finding a suitable initial guess is avoided by using homotopy methods such as fixed point and Newton's homotopy methods.

AIM AND OBJECTIVES

The aim of this research is to use homotopy continuation method to solve polynomial equations and has the following objectives:

(i) Applying the Newton's method in solving polynomial equation;

(ii) Using the phc (polynomial homotopy continuation) software to solve polynomial equation;

(iii)   Comparing solutions obtained from both Newton's method and the phc (polynomial homotopy continuation) software.

The homotopy continuation methods are rapidly being applied to numerous problems in engineering analysis like simulating and designing a chemical plant or optimizing a process, in

circuit board design, river routing, in motion path planning and also to mention but a few, in geographic information systems (GIS).

MAIN RESULT

We provide solutions to polynomial equations namely; the Legendre and Chebychev's polynomials using the Newton's method and the phc (general-purpose solver for polynomial systems by Homotopy continuation).

THE LEGENDRE POLYNOMIAL

We shall use the Newton's method in finding the roots of the Legendre polynomial and also, the phc

(polynomial homotopy continuation)

$$p(x) = \frac{1}{48}(693x^6 - 945x^4 + 315x^2 - 15)$$

Using the Newton's method:

$$p(x) = \frac{1}{48}(693x^6 - 945x^4 + 315x^2 - 15)$$

$$p'(x) = \frac{693}{8}x^5 - \frac{945}{12}x^3 + \frac{315}{24}x$$

$$x_{n+1} = x_n - \frac{p(x)}{p'(x)}$$

$$x_{n+1} = x_n - \frac{\left[\dfrac{693}{48}x_n^6 - \dfrac{945}{48}x_n^4 + \dfrac{315}{48}x_n^2 - \dfrac{15}{48}\right]}{\left[\dfrac{693}{8}x_n^5 - \dfrac{945}{12}x_n^3 + \dfrac{315}{24}x_n\right]}$$

$$x_{n+1} = x_n - \frac{(14.4375x_n^6 - 19.6875x_n^4 + 6.5625x_n^2 - 0.3125)}{86.625x_n^5 - 78.75x_n^3 + 13.125x}$$

$n = 0, 1, \ldots$

for $n = 0$,

$$x_1 = x_0 - \frac{(14.4375x_0^6 - 19.6875x_0^4 + 6.5625x_0^2 - 0.3125)}{86.625x_0^5 - 78.75x_0^3 + 13.125x_0}$$

Take $x_0 = 0.75$

$$x_1 = 0.75 - \frac{\left(14.4375(0.75)^6 - 19.6875(0.75)^4 + 6.5625(0.75)^2 - 0.3125\right)}{86.625(0.75)^5 - 78.75(0.75)^3 + 13.125(0.75)}$$

$$x_1 = 0.75 - \frac{(2.56956 - 6.22925 + 3.69 - 0.3125)}{20.55652 - 33.22266 + 9.84375}$$

$$x_1 = 0.75 - \frac{(-0.28219)}{(-2.82239)}$$

$$x_1 = 0.75 - \frac{0.28219}{2.82239}$$

$$x_1 = 0.75 - 0.09998$$

$$x_1 = 0.65002 (1^{st} \text{ iteration}).$$

for $n = 1$,

$$x_2 = x_1 - \frac{(14.4375x_1^6 - 19.6875x_1^4 + 6.5625x_1^2 - 0.3125)}{86.625x_1^5 - 78.75x_1^3 + 13.125x_1}$$

$$x_2 = 0.65002 - \frac{\left(14.4375(0.65002)^6 - 19.6875(0.65002)^4 + 6.5625(0.65002)^2 - 0.3125\right)}{86.625(0.65002)^5 - 78.75(0.65002)^3 + 13.125(0.65002)}$$

$$x_2 = 0.65002 - \frac{(1.08906 - 3.51477 + 2.77283 - 0.3125)}{10.05256 - 21.62872 + 8.53151}$$

$$x_2 = 0.65002 + \frac{0.03462}{3.04465}$$

$$x_2 = 0.65002 + 0.01137$$

$$x_2 = 0.66139 (2^{nd} \text{ iteration})$$

Hence, 0.66139 is a root of p(x).

Below is the solution of the legendre polynomial provided by the phc software:

1.44375000000000E+01*x^6-1.96875000000000E+01*x^4
+ 6.56250000000000E+00*x^2-3.12500000000000E-01;

ROOT COUNTS :

total degree : 6

HOMOTOPY PARAMETERS :

    d : 16

    k : 2

    a : -4.24751288571780E-01 -9.05310081053234E-01

    t : 1.00000000000000E+00   0.00000000000000E+00

    no projective transformation

***************** CURRENT CONTINUATION PARAMETERS *****************

GLOBAL MONITOR :

  1. the condition of the homotopy   : 0

  2. number of paths tracked simultaneously   : 1

  3. maximum number of steps along a path   : 500

  4. distance from target to start end game   : 1.000E-01

  5. order of extrapolator in end game   : 0

  6. maximum number of re-runs   : 1

STEP CONTROL (PREDICTOR) :   along path   : end game

  7: 8. type (x:Sec,t:Rea):(x:Sec,t:Rea) : 0   : 0

  9:10. minimum step size   : 1.000E-06   : 1.000E-08

11:12. maximum step size   : 1.000E-01   : 1.000E-02

13:14. reduction factor for step size   : 7.000E-01   : 5.000E-01

15:16. expansion factor for step size   : 1.250E+00   : 1.100E+00

17:18. expansion threshold   : 1   : 1

PATH CLOSENESS (CORRECTOR) :   along path   : end game

19:20. maximum number of iterations   : 4   : 4

21:22. relative precision for residuals   : 1.000E-09   : 1.000E-11

23:24. absolute precision for residuals   : 1.000E-09   : 1.000E-11

25:26. relative precision for corrections : 1.000E-09   : 1.000E-11

27:28. absolute precision for corrections : 1.000E-09   : 1.000E-11

SOLUTION TOLERANCES :   along path   : end game

29:30. inverse condition of Jacobian   : 1.000E-04   : 1.000E-12

31:32. clustering of solutions   : 1.000E-04   : 1.000E-12

33:34. solution at infinity   : 1.000E+08   : 1.000E+12

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OUTPUT INFORMATION DURING CONTINUATION :

 0 : no intermediate output information during continuation

TIMING INFORMATION for continuation

The elapsed time in seconds was   0.046000000 = 0h 0m 0s 46ms

THE SOLUTIONS :

6 1

===

solution 1 :   start residual : 2.220E-16   #iterations : 1   success

t : 1.00000000000000E+00   0.00000000000000E+00

m : 1

the solution for t :

 x : 9.32469514203152E-01 -1.36845553156720E-48

== err : 2.348E-17 = rco : 1.000E+00 = res : 2.220E-16 = real regular ==

solution 2 :   start residual : 0.000E+00 #iterations : 1   success

t : 1.00000000000000E+00   0.00000000000000E+00

m : 1

the solution for t :

 x : 2.38619186083197E-01   0.00000000000000E+00

== err : 0.000E+00 = rco : 1.000E+00 = res : 0.000E+00 = real regular ==

solution 3 :   start residual : 5.551E-17 #iterations : 1   success

t : 1.00000000000000E+00   0.00000000000000E+00

m : 1

the solution for t :

 x : -6.61209386466265E-01   1.55575381946529E-60

== err : 1.769E-17 = rco : 1.000E+00 = res : 5.551E-17 = real regular ==

solution 4 :   start residual : 2.220E-16 #iterations : 1   success

t : 1.00000000000000E+00   0.00000000000000E+00

m : 1

the solution for t :

 x : -9.32469514203152E-01   1.36845553156720E-48

== err : 2.348E-17 = rco : 1.000E+00 = res : 2.220E-16 = real regular ==

solution 5 :   start residual : 0.000E+00 #iterations : 1   success

t : 1.00000000000000E+00   0.00000000000000E+00

m : 1

the solution for t :

 x : -2.38619186083197E-01   0.00000000000000E+00

== err : 0.000E+00 = rco : 1.000E+00 = res : 0.000E+00 = real regular ==

solution 6 :   start residual : 5.551E-17 #iterations : 1   success

t : 1.00000000000000E+00   0.00000000000000E+00

m : 1

the solution for t :

 x : 6.61209386466265E-01   -1.55575381946529E-60

== err : 1.769E-17 = rco : 1.000E+00 = res : 5.551E-17 = real regular ==

===

A list of 6 solutions has been refined :

Number of regular solutions   : 6.

Number of singular solutions   : 0.

Number of real solutions   : 6.

Number of complex solutions   : 0.

Number of clustered solutions   : 0.

Number of solutions at infinity : 0.

Number of failures   : 0.

===

Frequency tables for correction, residual, and condition numbers :

FreqCorr : 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 6 : 6

FreqResi : 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 6 : 6

FreqCond :6 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 : 6

Small correction terms and residuals counted to the right.

Well conditioned and distinct roots counted to the left.

TIMING INFORMATION for Root Refining
The elapsed time in seconds was   0.016000000 = 0h 0m 0s 16ms

TIMING INFORMATION for solving the polynomial system
The elapsed time in seconds was   63.180000000 = 0h 1m 3s180ms

PHC ran from 25 March 2013, 09:03:38 till 25 March 2013, 09:09:11.
The total elapsed time is 333 seconds = 5 minutes 33 seconds.

CHEBYCHEV'S POLYNOMIAL

We shall use the Newton's method in finding the roots of the chebychev's method and also, the phc (polynomial homotopy continuation)

$$T(x) = 32x^6 - 48x^4 + 18x^2 - 1 = 0.$$

using Newton's method:

$$T(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T'(x) = 192x^5 - 192x^3 + 36x$$

$$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n)}$$

$$x_{n+1} = x_n - \frac{(32x_n^6 - 48x_n^4 + 18x_n^2 - 1)}{192x_n^5 - 192x_n^3 + 36x_n}$$

$$n = 0, 1, \ldots$$

for $n = 0$

$$x_1 = x_0 - \frac{(32x_0^6 - 48x_n^4 + 18x_0^2 - 1)}{192x_0^5 - 192x_0^3 + 36x_0}$$

Take $x_0 = 0.6$

$$x_1 = 0.6 - \frac{\left(32(0.6)^6 - 48(0.6)^4 + 18(0.6)^2 - 1\right)}{192(0.6)^5 - 192(0.6)^3 + 36(0.6)}$$

$$x_1 = 0.6 - \frac{(1.49299 - 6.2208 + 6.48 - 1)}{14.92992 - 41.472 + 21.6}$$

$$x_1 = 0.6 + \frac{0.75219}{4.94208}$$

$$x_1 = 0.6 + 0.1522$$

$$x_1 = 0.7522(1^{st} \text{ iteration})$$

$$\text{for } n = 0$$

$$x_2 = x_1 - \frac{(32x_1^6 - 48x_1^4 + 18x_1^2 - 1)}{192x_1^5 - 192x_1^3 + 36x_1}$$

$$x_2 = 0.7522 - \frac{(5.79629 - 15.36649 + 10.18449 - 1)}{46.23468 - 81.71489 + 27.0792}$$

$$x_2 = 0.7522 - \frac{0.38571}{8.40101}$$

$$x_2 = 0.7522 - 0.04591$$

$$x_2 = 0.70629(2^{nd} \text{ iteration})$$

$$\text{Hence, } 0.70629 \text{ is a root of } T(x).$$

Below is the solution of the chebychev's polynomial provided by the phc software:

32*x^6-48*x^4+18*x^2-1;

ROOT COUNTS :

total degree : 6

HOMOTOPY PARAMETERS :
 d : 16
 k : 2
 a : 8.10952587803780E-01   5.85111869931172E-01
 t : 1.00000000000000E+00   0.00000000000000E+00
 no projective transformation

***************** CURRENT CONTINUATION PARAMETERS *****************
GLOBAL MONITOR :
 1. the condition of the homotopy   : 0
 2. number of paths tracked simultaneously   : 1
 3. maximum number of steps along a path   : 500

  4. distance from target to start end game   : 1.000E-01
  5. order of extrapolator in end game   : 0
  6. maximum number of re-runs   : 1
  STEP CONTROL (PREDICTOR) :   along path   : end game
  7: 8. type (x:Sec,t:Rea):(x:Sec,t:Rea)   : 0   : 0
  9:10. minimum step size   : 1.000E-06   : 1.000E-08
  11:12. maximum step size   : 1.000E-01   : 1.000E-02
  13:14. reduction factor for step size   : 7.000E-01   : 5.000E-01
  15:16. expansion factor for step size   : 1.250E+00   : 1.100E+00
  17:18. expansion threshold   : 1   : 1
  PATH CLOSENESS (CORRECTOR) :   along path   : end game
  19:20. maximum number of iterations   : 4   : 4
  21:22. relative precision for residuals   : 1.000E-09   : 1.000E-11
  23:24. absolute precision for residuals   : 1.000E-09   : 1.000E-11
  25:26. relative precision for corrections   : 1.000E-09   : 1.000E-11
  27:28. absolute precision for corrections   : 1.000E-09   : 1.000E-11
  SOLUTION TOLERANCES :   along path   : end game
  29:30. inverse condition of Jacobian   : 1.000E-04   : 1.000E-12
  31:32. clustering of solutions   : 1.000E-04   : 1.000E-12
  33:34. solution at infinity   : 1.000E+08   : 1.000E+12
  ****

OUTPUT INFORMATION DURING CONTINUATION :
  0 : no intermediate output information during continuation

TIMING INFORMATION for continuation
The elapsed time in seconds was   0.016000000 = 0h 0m 0s 16ms

THE SOLUTIONS :

6 1
==
solution 1 :   start residual : 2.220E-16 #iterations : 1 success
t : 1.00000000000000E+00 0.00000000000000E+00
m : 1
the solution for t :
 x : 7.07106781186548E-01 -1.18694596821997E-65
== err : 2.617E-17 = rco : 1.000E+00 = res : 2.220E-16 = real regular ==
solution 2 :   start residual : 1.110E-16 #iterations : 1 success
t : 1.00000000000000E+00 0.00000000000000E+00
m : 1
the solution for t :
 x : -2.58819045102521E-01 0.00000000000000E+00
== err : 1.787E-17 = rco : 1.000E+00 = res : 1.110E-16 = real regular ==
solution 3 :   start residual : 1.443E-15 #iterations : 1 success
t : 1.00000000000000E+00 0.00000000000000E+00
m : 1
the solution for t :
 x : -9.65925826289068E-01 1.36845553156720E-48
== err : 6.226E-17 = rco : 1.000E+00 = res : 1.998E-15 = real regular ==
solution 4 :   start residual : 2.220E-16 #iterations : 1 success
t : 1.00000000000000E+00 0.00000000000000E+00
m : 1
the solution for t :
 x : -7.07106781186548E-01 5.93472984109987E-66
== err : 2.617E-17 = rco : 1.000E+00 = res : 2.220E-16 = real regular ==

solution 5 :   start residual : 1.110E-16   #iterations : 1   success
t : 1.00000000000000E+00   0.00000000000000E+00
m : 1
the solution for t :
 x : 2.58819045102521E-01   0.00000000000000E+00
== err : 1.787E-17 = rco : 1.000E+00 = res : 1.110E-16 = real regular ==
solution 6 :   start residual : 1.443E-15   #iterations : 1   success
t : 1.00000000000000E+00   0.00000000000000E+00
m : 1
the solution for t :
 x : 9.65925826289068E-01   -1.36845553156720E-48
== err : 6.226E-17 = rco : 1.000E+00 = res : 1.998E-15 = real regular ==
==
A list of 6 solutions has been refined :
Number of regular solutions   : 6.
Number of singular solutions   : 0.
Number of real solutions   : 6.
Number of complex solutions   : 0.
Number of clustered solutions   : 0.
Number of solutions at infinity   : 0.
Number of failures   : 0.
==
Frequency tables for correction, residual, and condition numbers :
FreqCorr : 0 0 0 0 0 0 0 0 0 0 0 0 0 0 6 : 6
FreqResi : 0 0 0 0 0 0 0 0 0 0 0 0 0 0 6 : 6
FreqCond :6 0 0 0 0 0 0 0 0 0 0 0 0 0 0 : 6
Small correction terms and residuals counted to the right.
Well conditioned and distinct roots counted to the left.

TIMING INFORMATION for Root Refining
The elapsed time in seconds was   0.000000000 = 0h 0m 0s 0ms

TIMING INFORMATION for solving the polynomial system
The elapsed time in seconds was   129.325000000 = 0h 2m 9s325ms

PHC ran from 25 March 2013, 05:16:33 till 25 March 2013, 05:20:34.
The total elapsed time is 240 seconds = 4 minutes

DISCUSSION

The roots of polynomial equations was obtained using both the Newton's method and the phc (polynomial homotopy continuation) and their result was compared, we observed that the Newton's method is locally convergent while, the homotopy continuation method is globally convergent that is, blowing out all isolated real roots of the polynomial equation.

CONCLUSION

Many iterative techniques and Newton method for the solution of non-linear equations have the drawback that the convergence depends on only good approximation.

If no approximation roots are known most of the classical method may be of little use. The method for following the path is a substantial global convergence. But Newton method could circle or could blow up when the Jacobian matrix at some point is singular. The imbedding method has the advantages of producing solutions over a large range of the independent variable, thereby giving a complete description of the solution behaviour and is use as a tool in overcoming the local convergence of iterative processes. The imbedding methods may be considered as a possibility to widen the domain of convergence or from another point of view, as a procedure to obtain sufficiently close starting points.

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