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Approximation of a Function f Belonging to Lip Class by (N, p, q)C_1 Means of its Fourier Series

Binod Prasad Dhakal
Central Department of Education (Mathematics), Tribhuvan University, Nepal

Abstract

An estimate for the degree of approximation of function $f \in Lip\alpha$ class by (N, p, q) C_1 means of Fourier series has been established.

1. Definitions

The Fourier series of 2π periodic Lebesgue integrable $f(t)$ over $[-\pi, \pi]$ is given by

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (1)$$

The degree of approximation $E_n(f)$ of a function f: R→ R by a trigonometric polynomial t_n of degree n is defined by (Zygmund [1])

$$E_n(f) = \|t_n - f\|_\infty = \sup.\{|t_n(x) - f(x)| : x \in \Re\}.$$

A function $f \in Lip\alpha$ if,
$|f(x+t) - f(x)| = O(|t|^\alpha)$ for $0 < \alpha \le 1$. (Dhakal [2])

Let $\sum_{m=0}^{\infty} u_n$ be an infinite series such that whose n^{th} partial sum $s_n = \sum_{k=0}^{n} u_k$. Write $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$ is (C,1) means of the sequence $\{S_n\}$. If $\sigma_n \to S$, as $n \to \infty$ then the sequence $\{S_n\}$ is said to be summable by Cesàro method (C,1) to S.

The generalized Nörlund transform (N, p, q) of the sequence $\{S_n\}$ is the sequence $\{t_n^{p,q}\}$ where $t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} S_{n-k}$. If $t_n^{p,q} \to S$ as $n \to \infty$ then the sequence $\{S_n\}$ is said to be summable by generalized Nörlund method (N, p, q) to S (Borwein [3]).

The (N, p, q) transform of the (C,1) transform defines the (N, p, q)C_1 transform $\{t_n^{p,q,c_1}\}$ of the partial sum $\{S_n\}$ of the series $\sum_{n=0}^{\infty} u_n$ .Thus,

$t_n^{p,q,c_1} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\sigma_{n-k} \to S$, as $n \to \infty$ then the sequence $\{S_n\}$ is said to be summable by (N, p, q)C_1 method to S..

Some important particular cases of (N, p, q)C_1 means are:

(i) $(N, p_n)C_1$ if $q_n = 1\ \forall n$.

(ii) $(\overline{N}, q_n)C_1$ if $p_n = 1 \forall n$.

(iii) $(C,\delta)C_1$ if $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$ and $q_n = 1 \forall n$.

We shall use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x) \qquad (2)$$

$$(NC)_n(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \qquad (3)$$

2. Theorem

Theorem. If $f : R \to R$ is 2π periodic, Lebesgue integrable over $[-\pi,\pi]$ and Lip α class function, then the degree of approximation of function f by (N, p, q) C_1 summability means, $t_n^{p,q,c_1} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\sigma_{n-k}$ of the Fourier series (1) is given by, for n = 0, 1, 2 …,

$$\|t_n^{p,q,c_1} - f\|_\infty = O\left(\frac{\log(n+1)\pi e}{(n+1)^\alpha}\right) \text{for } 0 < \alpha \le 1, \qquad (4)$$

provide $\{p_n\}$ and $\{q_n\}$ are two sequences of positive real constants of regular generalized Nörlund method (N, p, q) such that

$$\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1} = O\left(\frac{R_n}{n+1}\right)\ \forall\ n \ge 0. \qquad (5)$$

3. Proof of the Theorem

Following Titchmarsh [4], n^{th} partial sum $S_n(x)$ of the Fourier series (1) at t = x ∈ [-π, π] is given by

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt.$$

The (C,1) transform i.e. σ_n of S_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}(S_k(x) - f(x)) = \frac{1}{2(n+1)\pi}\int_0^\pi \frac{\phi(t)}{\sin\frac{t}{2}}\sum_{k=1}^{n}\sin\left(k+\frac{1}{2}\right)t\,dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^\pi \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\phi(t)\,dt.$$

Denoting (N, p, q) transform of σ_n i.e. (N, p, q)C_1 transform of S_n by t_n^{p,q,c_1}, we have

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\{\sigma_{n-k}(x) - f(x)\} = \int_0^\pi \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\phi(t)\,dt$$

$$t_n^{p,q,c_1}(x) - f(x) = \int_0^\pi (NC)_n(t)\,\phi(t)$$

$$= \int_0^{\frac{1}{n+1}} (NC)_n(t)\phi(t)\,dt + \int_{\frac{1}{n+1}}^{\pi} (NC)_n(t)\phi(t)\,dt$$

$= I_1 + I_2$ say. (6)

For I_1 and $0 < t \le \frac{1}{n+1}$

$$(NC)_n(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{n-k+1}(n-k+1)^2\frac{\sin^2\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\left(\text{Since } \sin n\theta\le n\sin\theta\le n\theta \text{ for } 0<\theta<\frac{1}{n}\right)$$

$$=\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,(n-k+1)$$

$$\le\frac{n+1}{2\pi R_n}\sum_{k=0}^{n}p_k\,q_{n-k}$$

$$=\frac{n+1}{2\pi}$$

$$=O(n+1). \qquad (7)$$

Since, $|f(x+t)-f(x)|=O(|t|^{\alpha})$ for $0<\alpha\le 1$,

if $f\in Lip\alpha$.

We have, $\phi(t)=f(x+t)+f(x-t)-2f(x)$

$$=[f(x+t)-f(x)]+[f(x-t)-f(x)]$$

$$=O(t^{\alpha})+O(t^{\alpha})$$

$$=O(t^{\alpha}). \qquad (8)$$

Now, using (7) and (8) and the fact that $\phi(t)\in Lip\alpha$, we have

$$|I_1|\le\int_0^{\frac{1}{n+1}}|(NC)_n(t)|\,|\phi(t)|\,dt$$

$$=\int_0^{\frac{1}{n+1}}O(n+1)\,O(t^{\alpha})\,dt$$

$$=O(n+1)\left[\int_0^{\frac{1}{n+1}}t^{\alpha}\,dt\right]$$

$$=O(n+1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$$

$$=O(n+1)\frac{1}{(\alpha+1)(n+1)^{\alpha+1}}$$

$$=O\left(\frac{1}{(n+1)^{\alpha}}\right). \qquad (9)$$

For I_2 and $\frac{1}{n+1}<t<\pi$

$$(NC)_n(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{n-k+1}\frac{\pi^2}{t^2}, \quad \text{by Jordan's Lemma}$$

$$=\frac{\pi}{2R_nt^2}\sum_{k=0}^{n}\frac{p_k\,q_{n-k}}{n-k+1}$$

$$=\frac{\pi}{2R_nt^2}O\left(\frac{R_n}{n+1}\right), \text{ by the hypothesis of the theorem}$$

$$=O\left(\frac{1}{(n+1)t^2}\right). \qquad (10)$$

Using (8) and (10), we have

$$|I_2|\le\int_{\frac{1}{n+1}}^{\pi}|(NC)_n(t)|\,|\phi(t)|\,dt$$

$$=\int_{\frac{1}{n+1}}^{\pi}O\left(\frac{1}{(n+1)t^2}\right)O(t^{\alpha})\,dt$$

$$=O\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi}t^{\alpha-2}\,dt$$

$$=\begin{cases}O\left(\frac{1}{n+1}\right)\left[\frac{t^{\alpha-1}}{\alpha-1}\right]_{\frac{1}{n+1}}^{\pi}, & \text{for } 0<\alpha<1\\ O\left(\frac{1}{n+1}\right)[\log t]_{\frac{1}{n+1}}^{\pi}, & \text{for } \alpha=1\end{cases}$$

$$=\begin{cases}O\left(\frac{1}{n+1}\right)\left(\frac{1}{\alpha-1}\right)\left[\pi^{\alpha-1}-\frac{1}{(n+1)^{\alpha-1}}\right], & \text{for } 0<\alpha<1\\ O\left(\frac{1}{n+1}\right)\left[\log\pi-\log\left(\frac{1}{n+1}\right)\right], & \text{for } \alpha=1\end{cases}$$

$$\le\begin{cases}O\left(\frac{1}{\alpha-1}\right)\left[\frac{\pi^{\alpha-1}}{n+1}+\frac{1}{(n+1)^{\alpha}}\right], & \text{for } 0<\alpha<1\\ O\left(\frac{1}{n+1}\right)[\log(n+1)\pi], & \text{for } \alpha=1\end{cases}$$

$$\le\begin{cases}O\left(\frac{1}{\alpha-1}\right)\left[\frac{\pi^{\alpha-1}+1}{(n+1)^{\alpha}}\right], & \text{for } 0<\alpha<1\\ O\left[\frac{\log(n+1)\pi}{(n+1)}\right], & \text{for } \alpha=1\end{cases}$$

$$=\begin{cases}O\left(\frac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1\\ O\left[\frac{\log(n+1)\pi}{(n+1)}\right], & \text{for } \alpha=1.\end{cases} \qquad (11)$$

Collecting (6), (9), (11); we have

$$\left|t_n^{p,q,c_1}(x)-f(x)\right|=\begin{cases}O\left(\frac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1\\ O\left(\frac{1}{n+1}\right)+O\left(\frac{\log(n+1)\pi}{(n+1)}\right), & \text{for } \alpha=1\end{cases}$$

$$= \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right), & \text{for } \alpha=1 \end{cases}$$

$$\leq \begin{cases} O\left(\frac{\log(n+1)\pi e}{(n+1)^{\alpha}}\right), & \text{for } 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi e}{n+1}\right), & \text{for } \alpha=1 \end{cases}$$

$$= O\left(\frac{\log(n+1)\pi e}{(n+1)^{\alpha}}\right), \quad \text{for } 0<\alpha\leq 1$$

Hence,

$$\left\| t_n^{p,q,c_1} - f \right\|_{\infty} = \sup\left\{ \left| t_n^{p,q,c_1}(x) - f(x) \right| : x \in R \right\}$$

$$= O\left(\frac{\log(n+1)\pi e}{(n+1)^{\alpha}}\right), \qquad 0 < \alpha \leq 1.$$

Thus, the theorem is completely established.

4. References

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