International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 1 Issue 9, November- 2012

Argument Estimates Of Strongly Close-to-star Functions In A Sector

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October 30, 2012

Abstract

In the present investigation, we obtain some sufficient condition for a normalized strongly close-to-star functions in the open disk $\mathbb{U} = \{z \in C : |z| < 1\}$ to satisfy the condition

$$-\frac{\pi}{2}\beta \leq arg\left\{\frac{f(z)}{g(z)}\right\} \leq \frac{\pi}{2}\alpha, \quad 0 \leq \alpha,\ \beta \leq 1.$$

The aim of this paper is to generalize a result obtained by N.E.Cho and S.Owa.

2010 AMS Subject Classification: Primary 30C45.

Key words and Phrases: Analytic functions,Strongly Close-to-Star functions,convex functions,Starlike functions.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}, \tag{1.1}$$

which are *analytic* in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions. Let us denote $\mathcal{S}^*, \mathcal{K}$ *and* $\mathcal{C}$ be the subclasses of $\mathcal{A}$, consisting of functions which are respectively starlike,convex and close-to-convex in $\mathbb{U}$.

Let $f(z)$ and $g(z)$ be analytic functions in $\mathbb{U}$. We say that $f(z)$ is subordinate to $g(z)$ if there exist analytic function $w(z)$ such that $w(0) = 0$,$|w(z)| < 1$ with $f(z) = g(w(z))$ and is denoted by $f \prec g$.

Let

$$\mathcal{S}^*[A, B] = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{U} \ \ -1 \leq B < A \leq 1\right\}$$

and

$$\mathcal{K}[A, B] = \left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{U} \ \ -1 \leq B < A \leq 1\right\}$$

The class $\mathcal{S}^*[A, B]$ and related classes were studied by Janowski[1] and Silverman and Silvia [4] proved that a function $f(z)$ is in $\mathcal{S}^*[A,B]$ iff

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (z \in \mathbb{U}; B \neq -1) \tag{1.2}$$

$$and \quad Re\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1-A}{2} \quad (z \in \mathbb{U}; B = -1) \tag{1.3}$$

Lemma 1.1. *[3] Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$. If there exists two points $z_1, z_2 \in \mathbb{U}$ such that $|z_1| = |z_2|$*

$$-\frac{\pi}{2}\beta = arg\, p(z_1) < arg\, p(z) < arg\, p(z_2) = \frac{\pi}{2}\alpha, \quad \alpha,\ \beta > 0 \ \ and\,, for|z| < |z_1| = |z_2|,$$

then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = i\left(\frac{\alpha+\beta}{2}\right)m$$
$$and$$
$$\frac{z_2 p'(z_2)}{p(z_2)} = i\left(\frac{\alpha+\beta}{2}\right)m$$
$$where \quad m \geq \frac{1-|\delta|}{1+|\delta|} \quad and \quad \delta = itan\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

Theorem 1.1. *Let $f \in \mathcal{A}$. If*

$$\left|arg\left\{\left(\frac{f'(z)}{g'(z)}\right)^a \left(\frac{f(z)}{g(z)}\right)^b\right\}\right| \leq \frac{\pi}{2}\delta$$

for some

$$g(z) \in \mathcal{K}[A,B],$$

then

$$\left|arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\pi}{2}\alpha$$

where $\alpha (0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} (a+b)\alpha + \frac{2}{\pi}atan^{-1}\dfrac{m\alpha sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + m\alpha cos\frac{\pi}{2}(1-t(A,B))} & , B \neq -1 \\ (a+b)\alpha & , B = -1 \end{cases}$$

where $t(A,B) = \dfrac{2}{\pi}sin^{-1}\left(\dfrac{A-B}{1-AB}\right)$.

Proof. Let $p(z) = \dfrac{f(z)}{g(z)}$, $q(z) = \dfrac{zg'(z)}{g(z)}$
by differentiating logarithmically, we have

$$\frac{p'(z)}{p(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}$$

A simple computation shows that

$$\left(\frac{f'(z)}{g'(z)}\right)^a \left(\frac{f(z)}{g(z)}\right)^b = (p(z))^a + b\left(1 + \frac{1}{q(z)}\frac{zp'(z)}{p(z)}\right)^a$$

Since $g(z) \in \mathcal{K}[A,B]$, $\ g(z) \in \mathcal{S}^*[A,B]$.

If we take $q(z) = \rho e^{i\frac{\pi}{2}\phi}$, $z \in \mathbb{U}$, then it follows from(1.2) and (1.3) that

$$\frac{1-A}{1-B} < \rho < \frac{1+A}{1+B}, \quad -t(A,B) < \phi < t(A,B), if B \neq -1,$$
$$and \frac{1-A}{2} < \rho < \infty, \quad -1 < \phi < \infty, if B = -1,$$
$$where \quad t(A,B) = \frac{2}{\pi} sin^{-1}\left(\frac{A-B}{1-AB}\right).$$

Let $p(z) = \frac{f(z)}{g(z)}$, $f \in \mathcal{A}$ and $g \in \mathcal{A}$. If there exists two points $z_1, z_2 \in \mathbb{U}$ such that

$$-\frac{\pi}{2}\beta = arg\, p(z_1) < arg\, p(z) < arg\, p(z_2) = \frac{\pi}{2}\alpha, \quad \alpha,\ \beta > 0\ \ and\, , for |z| < |z_1| = |z_2|,$$

then by lemma(1.1), we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\left(\frac{\alpha+\beta}{4}\right)\left(\frac{1+t_1^2}{t_1}\right)m$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\left(\frac{\alpha+\beta}{4}\right)\left(\frac{1+t_2^2}{t_2}\right)m. \tag{1.4}$$

where

$$e^{-i\frac{\pi}{2}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)}(p(z_1))^{\left(\frac{2}{\alpha+\beta}\right)} = -it_1$$

and

$$e^{-i\frac{\pi}{2}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)}(p(z_2))^{\left(\frac{2}{\alpha+\beta}\right)} = it_2, \quad t_1, t_2 > 0. \tag{1.5}$$

and

$$m \geq \frac{1-|\delta|}{1+|\delta|} \tag{1.6}$$

Let us put $z = z_2$. Then from (1.4),(1.5)and (1.6), we have

$$\begin{aligned}
arg\left\{\left(\frac{f'(z_2)}{g'(z_2)}\right)^a\left(\frac{f(z_2)}{g(z_2)}\right)^b\right\} &= (a+b)arg p(z_2) + aarg\left\{1 + \frac{1}{q(z_2)}\frac{z_2 p'(z_2)}{p(z_2)}\right\} \\
&= (a+b)\frac{\pi}{2}\alpha + a\ arg\left(1 + \frac{e^{-i\frac{\pi}{2}\phi}}{\rho} i\left(\frac{\alpha+\beta}{4}\right)\left(\frac{1}{t_2} + t_2\right)m\right) \\
&= \frac{\pi}{2}\alpha(a+b) + a\ arg\left(\rho + me^{i\frac{\pi}{2}(1-\phi)}\left(\frac{\alpha+\beta}{4}\right)\left(t_2 + \frac{1}{t_2}\right)\right) \\
&= \frac{\pi}{2}\alpha(a+b) + a\ arg\left(\rho + m\left(\frac{\alpha+\beta}{4}\right)\left(t_2 + \frac{1}{t_2}\right)\left(cos\frac{\pi}{2}(1-\phi) + isin\frac{\pi}{2}(1-\phi)\right)\right) \\
&\geq \frac{\pi}{2}\alpha(a+b) + a\ tan^{-1}\left\{\frac{m\left(\frac{\alpha+\beta}{4}\right)\left(t_2+\frac{1}{t_2}\right)sin\frac{\pi}{2}(1-\phi)}{\rho + m\left(\frac{\alpha+\beta}{4}\right)\left(t_2+\frac{1}{t_2}\right)cos\frac{\pi}{2}(1-\phi)}\right\}
\end{aligned}$$

Let us take $g(x) = x + \frac{1}{x}$, $x > 0$. Then attains the minimum value at $x = 1$. Therefore, we have

$$\begin{aligned} arg\left\{\left(\frac{f'(z_2)}{g'(z_2)}\right)^a \left(\frac{f(z_2)}{g(z_2)}\right)^b\right\} &\geq \frac{\pi}{2}\alpha(a+b) + atan^{-1}\left\{\frac{m\left(\frac{\alpha+\beta}{2}\right) sin\frac{\pi}{2}(1-\phi)}{\rho + m\left(\frac{\alpha+\beta}{2}\right) cos\frac{\pi}{2}(1-\phi)}\right\} \\ &\geq \frac{\pi}{2}\alpha(a+b) + atan^{-1}\left\{\frac{m\left(\frac{\alpha+\beta}{2}\right) sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + m\left(\frac{\alpha+\beta}{2}\right) cos\frac{\pi}{2}(1-t(A,B))}\right\} \\ &= \frac{\pi}{2}\delta \end{aligned}$$

where

$$\delta = \begin{cases} (a+b)\alpha + \frac{2}{\pi}atan^{-1}\left[\frac{m\alpha sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + m\alpha cos\frac{\pi}{2}(1-t(A,B))}\right] & ,B \neq -1 \\ (a+b)\alpha & ,B = -1 \end{cases},$$

$$t(A,B) = \frac{2}{\pi}sin^{-1}\left(\frac{A-B}{1-AB}\right),$$

$$m = \frac{1-|\delta|}{1+|\delta|},\ and\ \delta = itan\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

This contradicts the assumption of the theorem. For the case $z = z_1$, applying the same method as above, we have

$$arg\left\{\left(\frac{f'(z_1)}{g'(z_1)}\right)^a \left(\frac{f(z_1)}{g(z_1)}\right)^b\right\} \leq -\frac{\pi}{2}\beta(a+b) - atan^{-1}\left\{\frac{m\left(\frac{\alpha+\beta}{2}\right) sin\frac{\pi}{2}(1-\phi)}{\rho + m\left(\frac{\alpha+\beta}{2}\right) cos\frac{\pi}{2}(1-\phi)}\right\}$$

This contradiction completes the proof of the theorem. □

Taking $\alpha = \beta = 1$ in theorem (1.1), we have the result obtained by NAK Euncho and Shigeyoshi owa [2] By setting $a = 1, b = 0, \delta = 1, A = 1\ \ and B = -1$ in theorem (1.1), we have

Corollary 1.1. *Every close- to- convex function is close-to-star in* $\mathbb{U}$. *ie,*

$$\left|arg\left(\frac{f'(z)}{g'(z)}\right)\right| < \frac{\pi}{2}$$

ie,

$$Re\left(\frac{f'(z)}{g'(z)}\right) \geq 0$$

$$or$$

$$Re\left(\frac{f'(z)}{g'(z)}\right) \prec \frac{1+z}{1-z}.$$

If we put $g(z) = z$ in theorem (1.1), then by letting $B \to A(A < 1)$, we obtain

Corollary 1.2. *If* $f \in \mathcal{A}$ *and*

$$\left|arg\left\{(f'(z))^a \left(\frac{f(z}{z}\right)^b\right\}\right| < \frac{\pi}{2}\delta\ \ (a > 0, b\epsilon\mathbb{R}, 0 < \delta \leq 1)$$

then

$$|argf'(z)| < \frac{\pi}{2}\delta$$

where $\alpha (0 < \alpha \leq 1)$ *is the solution of the equation:*

$$\delta = (a+b)\alpha + \frac{2}{\pi} a \ tan^{-1}(\alpha).$$

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