Associated Sequence And A Generalized Common Fixed Point Theorem Under A New Condition

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ABSTRACT :

The purpose of this paper is to prove a generalized common fixed point theorem by changing the condition, used by V. Srinivas and B.V.B. Reddy [9]. To prove this theorem we use the definition of weakly compatible mapping & associated sequence.

Keywords :

Fixed Point, Self maps, compatible mappings, weakly compatible mappings, Cauchy sequence, associated sequence.

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Introduction :

G. Jungck [3] introduced the concept of compatible maps which is weaker than weakly commuting mappings. After words Jungck and Phoades [5] defined weaker class of maps known as weakly compatible maps. Further Srinivas and B.V.B. Reddy [9] used the concept of associated sequence.

1.1 Definition and Preliminaries :

1.1.1 Compatible Mappings :

If (X,d) be a metric space. Then two self maps A and B of (X,d) are said to be compatible mappings if $\lim_{n\to\infty}$ d (ABx_n, BAx_n) =0, whenever $\{x_n\}^{\infty}_{n=1}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = u,$ for some $u \in X$.

1.1.2 Weakly Compatible

Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be weakly compatible if they commute at their coincident point i.e. Ax = Bx , for some $x \in X$

$\Rightarrow$ ABx = BAx

It is clear that every compatible pair is weakly compatible but its converse need not be true.

1.1.3 Cauchy sequence

A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is called Cauchy sequence if for given $\in > 0$, there corresponds $n_0 \in N$ such that for all m, n $\geq n_0$, we have

$$d(x_{m,} x_n) < \in$$

1.1.4 Associated sequence

Suppose A, B, S and T are self maps of a metric space (X,d) satisfying the following condition

A (X) $\subseteq$ T(X) and B (X) $\subseteq$ S(X) ………………(1.1.4.1)

Then for an arbitrary $x_\circ \in X$ such that $A\,x_\circ = Tx_1$ and for this point x_1, there exists a point x_2 in X such that $Bx_1 = Sx_2$ and so on. Proceeding in similar manner, we can define a sequence $\{y_n\}_{n=1}^{\infty}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n} = Bx_{2n-1} = Sx_{2n} \text{ for } n \geq 0 \text{ ---------------- (1.1.4.2)}$$

Then this sequence is called "Associated sequence of $x_\circ$" relative to the four self maps A,B,S and T.

1.1.5 In (1998) Brijendra Singh and M.S. Chauhan [1] proved that common fixed point theorem for self maps A,B,S and T in metric space (X,d) by using the condition (1.1.4.1) and

$$\begin{aligned}[d\,(Ax, By)]^2 \leq k_1[d(Ax, Sx)\,d\,(By, Ty) + d\,(By, Sx)\,d\,(Ax, Ty)] \\ + k_2[d(Ax, Sx)\,d\,(Ax, Ty) + (By, Ty)\,d\,(By, Sx)]\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.1.5.1)$$

Where $0 \leq K_1 + 2K_2 < 1,\ K_1, K_2 \geq 0$

1.1.6 In (2012) V. Srinivas and B.V. B Reddy[9] established a generalize common fixed point theorem by using weakly compatible mapping and Associated sequence under the condition (1.1.4.1) and (1.1.4.2).

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Now we generalize the theorem using new condition under weakly compatible mapping and associated sequence.

Now we prove a lemma which plays an important role in our main theorem.

1.1.7 **Lemma:** Let A, B, S and T be a self mapping from a complete metric space (X,d) into itself satisfying the following conditions

$A(X) \subseteq T(X)\, and\, B(X) \subseteq S(X)$ --------- (1.1.8)

One of A, B, S or T is continuous such that

$$[d(Ax,By]^2 \le \alpha.\max\begin{pmatrix} d(Ax,Sx).\ d(By,Ty), \\ d(By,Sx).d(Ax,Ty), \\ d(Ax,Sx).d(Ax,Ty) \end{pmatrix} \text{..........................(1.1.9)}$$

Then the associated sequence $\{y_n\}_{n=1}^{\infty}$ relative to four self maps is a Cauchy sequence in X.

Proof: From conditions (1.1.8) & (1.1.9) and from the definition of associated sequence, we have

$$[d(y_{2n+1},y_{2n})]^2 = [d(Ax_{2n},Bx_{2n-1})]^2, \qquad By\ (1.1.4.2)$$

$$\le \alpha.\max\begin{pmatrix} d(Ax_{2n},Sx_{2n})\,d(Bx_{2n-1},Tx_{2n-1}), \\ d(Bx_{2n-1},Sx_{2n})\,d(Ax_{2n},Tx_{2n-1}), \\ d(Ax_{2n},Sx_{2n})\,d(Ax_{2n},Tx_{2n-1}), \end{pmatrix} \text{.......................................} By\,(1.1.9)$$

$$= \alpha.\max\begin{pmatrix} d(y_{2n+1},y_{2n})\,d(y_{2n},y_{2n-1}), \\ d(y_{2n},y_{2n})\,d(y_{2n+1},y_{2n-1}), \\ d(y_{2n+1},y_{2n})\,d(y_{2n+1},y_{2n-1}) \end{pmatrix}$$

$$= \alpha.\max\begin{pmatrix} d(y_{2n+1},y_{2n})\,d(y_{2n},y_{2n-1}), \\ 0, \\ d(y_{2n+1},y_{2n})\,d(y_{2n+1},y_{2n-1}) \end{pmatrix}$$

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$$=> [d(y_{2n+1}, y_{2n})]^2 \leq \alpha.\max\begin{pmatrix} d(y_{2n+1}, y_{2n}).d(y_{2n}, y_{2n-1}), \\ 0, \\ d(y_{2n+1}, y_{2n})\ d(y_{2n+1}, y_{2n-1}) \end{pmatrix}$$

$$=> [d(y_{2n+1}, y_{2n})] \leq \alpha.\max\begin{pmatrix} d(y_{2n+1}, y_{2n-1}), \\ 0, \\ (y_{2n+1}, y_{2n-1}) \end{pmatrix}$$

$$=> d(y_{2n+1}, y_{2n}) \leq \alpha.\max\begin{pmatrix} d(y_{2n}, y_{2n-1}), \\ 0, \\ d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) \end{pmatrix}$$

$\because$ By triangular Inequality

$d(y_{2n+1}, y_{2n-1}) \leq d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})$

$$=> d(y_{2n+1}, y_{2n}) \leq \alpha[d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]$$

$$=> d(y_{2n+1}, y_{2n}) - \alpha\, d(y_{2n+1}, y_{2n}) \leq \alpha\, d(y_{2n}, y_{2n-1})$$

$$=> (1-\alpha)\, d(y_{2n+1}, y_{2n}) \leq \alpha\, d(y_{2n}, y_{2n-1})$$

$$=> d(y_{2n+1}, y_{2n}) \leq \frac{\alpha}{1-\alpha}\, d(y_{2n}, y_{2n-1})$$

$$=> d(y_{2n+1}, y_{2n}) \leq \beta\, d(y_{2n}, y_{2n-1})$$

Where $\beta = \frac{\alpha}{1-\alpha} < 1 \quad \because\ \alpha < 1 => 0 < (1-\alpha) < 1$

Now $d(y_{2n}, y_{2n+1}) \leq \beta\ (y_{2n-1}, y_{2n})$

Then $d(y_n, y_{n+1}) \leq \beta\ (y_{n-1}, y_n)$

$\leq \beta^2\, d(y_{n-2}, y_{n-1})$

$\leq \beta^3\, d(y_{n-3}, y_{n-2})$

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$$\leq \beta^n d(y_0, y_1)$$

$$Now \quad d(y_{n+1}, y_{n+2}) \leq \beta d(y_n, y_{n+1})$$

$$\leq \beta^2 d(y_{n-1}, y_n)$$

$$\leq \beta^2 d(y_{n-2}, y_{n-3})$$

……………………………………………………..

……………………………………………………..

$$\leq \beta^{n+1} d(y_{0,}, y_1)$$

Similarly we can show

$$d(y_{n+2,}, y_{n+3}) \leq \beta^{n+2} d(_0, y_1)$$

$$d(y_{n+3,}, y_{n+4}) \leq \beta^{n+3} d(y_0, y_1)$$

……………………………………………………..

……………………………………………………..

$$d(y_{n+p-1,}, y_{n+p}) \leq \beta^{n+p-1} d(y_0, y_1) \quad , \textit{for every integer } p > 0$$

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$$Now \ d(y_{n,}, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1,}, y_{n+2}) + d\ (y_{n+2}, y_{n+3})$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + d\ (y_{n+p-1}, y_{n+1})$$

$$\leq \beta^n d(y_0, y_1) + \beta^{n+1} d(y_0, y_1) + \beta^{n+2} d(y_0, y_1) + \ldots\ldots\ldots + \beta^{n+p-1} d(y_0, y_1)$$

$$= [\beta^n + \beta^{n+1} + \beta^{n+2} + \ldots\ldots\ldots\ldots + \beta^{n+p-1}] d(y_0, y_1)$$

$$= \beta^n [1 + \beta + \beta^2 + \ldots\ldots\ldots\ldots + \beta^{n+p}] d(y_0, y_1)$$

$$\because \beta < 1 \Rightarrow \beta^n \to 0 \quad , \quad as \ n \to \infty$$

$$\Rightarrow d(y_n, y_{n+p}) \to 0, \quad as\, n \to \infty \quad , \textit{ for every integer } p > 0$$

This shows that $\{y_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* X.

$\because X$ *is a complete metric space.*

$=> \{y_n\}_{n=1}^{\infty}$ converges to z є X

Theorem :

Let A, B, S and T are self maps of a metric space (X,d) satisfying the condition (1.1.8)&(1.1.9)

And the pairs (A,S) and (B,T) are weakly compatible.

Further,

The associated sequence relative to self maps A, B, S and T such that the sequence

$Ax_0, Bx_1, Ax_2, Bx_3, ..., Ax_{2n}, Bx_{2n+1}, ...Convergs\ to\ z \in X\ as\ n \to \infty.$
Then A, B, S and T have a unique common fixed point z in X.

Proof –

$\because B(X) \subset S(X)$

ν є X such that z=Sν ………………………………..(1.1.10)

We have to prove Aν = Sν

Now consider

$$[d(Av, Bx_{2n+1})]^2 \le \alpha.\max\begin{pmatrix} d(Av, Sv).d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Bx_{2n+1}, Sv).d(Av, Tx_{2n+1}), \\ d(Av, Sv)\, d(Av, Tx_{2n+1}) \end{pmatrix}$$

$\because as\ n \to \infty\ ,\ Bx_{2n+1},\ Tx_{2n+1} \to z$

$$[d(Av, z)]^2 \le \alpha.\max\begin{pmatrix} d(Av, z)\ d(z, z), \\ d(z, z)\ d(Av, z), \\ d(Av, z)\, d(Av, z) \end{pmatrix}$$

$$= \alpha.\max\{0,\ 0, [d(Av, z)]^2\}$$
$$= \alpha[d(Av, z)]^2$$

$=> (d\,(Av, z)]^2 - \alpha[d(Av, z)]^2 \le 0$

$=> (d\,(Av, z)]^2\ (1-\alpha) \le 0$

$\because 0 \le \alpha < 1$

$=> 0 < 1-\alpha < 1$

$\therefore d(Av, z) = 0$

$=> Av = z$

$=> Av = Sv = z, \quad By\, using \ (1.1.10)$
......................................(1.1.11)

$\because The\ pair\ (A,S)\ is\ weakly\ Compatible.$

$=> ASv = SAv$

$=> Az = Sz \quad By\ (1.1.11)$
....(1.1.12)

$Now \ \because \ A(X) \subset T(X)$

$=> \exists\ \omega \in X\ \ such\, that\ z = Tw$.....................................(1.1.13)

Now we prove $B\omega = T\omega$

Now consider

$$[d(Av, B\omega)]^2 \le \alpha.\max\begin{pmatrix} d(Av, Sv)\, d(B\omega, T\omega), \\ d(B\omega, Sv)\ d(Av, T\omega), \\ d(Av, Sv)\, d(Av, T\omega) \end{pmatrix}, \quad By \ (1.1.9)$$

$$=> [d(z, B\omega)]^2 \le \alpha.\max\begin{pmatrix} d(z,z)\ d(B\omega, z), \\ d(B\omega, z)\ d(z,z), \\ d(z,z)\, d(z,z) \end{pmatrix}, \quad By \ (1.1.11)\ \&\ (1.1.13)$$

$=> [d(z, b\omega)]^2 \le 0$

But square of distance may not be less than zero.

$=> [d(z, B\omega]^2\ \ must\, be\, equal\, to\, Zero.$

$we\, get\, (d\,(z, B\omega]^2 = 0$

$=> d\,(z, B\omega] = 0$

$=> z = B\omega$..(1.1.14)

By (1.1.13) & (1.1.14)

$B\omega = T\omega = z$ ....(1.1.15)

Again since the pair (B,T) is weakly compatible.

$\Rightarrow Bz = Tz,$ By (1.1.15)
..................................(1.1.16)

Now Consider

$$[d(Az,z)]^2 = (d(Az,B\omega))^2 \quad , \quad By \ (1.1.15)$$

$$= \alpha.\max\begin{pmatrix} d(Az,Sz)\ d(B\omega,T\omega), \\ d(B\omega,Sz)\ d(Az,T\omega), \\ d(Az,Sz)\,d(Az,T\omega) \end{pmatrix}, \quad By \ (1.1.9)$$

$$= \alpha.\max\begin{pmatrix} d(Az,Az)\ d(Z,Z), \\ d(Z,Az)\ d(Az,Z), \\ d(Az,Az)\,d(Az,z) \end{pmatrix}, \quad By \ (1.1.12) \ \& \ (1.1.15)$$

$$= \alpha.\ d(Az,z)]^2$$

$$\Rightarrow [d(Az,Az)]^2 \le \alpha[d(Az,z)]^2$$

$$\Rightarrow [d(Az,z)]^2(1-\alpha) \le 0$$

$$\Rightarrow [d(Az,z)]^2 = 0 \qquad \because \alpha < 1 \quad \Rightarrow \quad 0 < (1-\alpha)$$

$$\Rightarrow [d(Az,z) = 0$$

$\Rightarrow Az = z$ ....$(1.1.17)$

$\Rightarrow Az = Sz = z$, By (1.1.12) & $(1.1.17)$
.... (1.1.18)

Again consider

$$[d(z,Bz)]^2 = (d(Av,Bz))^2$$

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$$\leq \alpha.\max\begin{pmatrix} d(Av,Sv)\ d(Bz,Tz), \\ d(Bz,Sv)\ d(Av,Tz), \\ d(Av,Sv)\, d(Av,Tz) \end{pmatrix}, \quad By \quad (1.1.19)$$

$$= \alpha.\max\begin{pmatrix} d(z,z)\ d(Bz,Bz), \\ d(Bz,z)\ d(z,Bz), \\ d(z,z)\, d(z,Bz) \end{pmatrix}$$

$$\because \ Av = Sv = Bw = Tw = z \ and \ Bz = Tz$$

$$=> [d(z,Bz)]^2 \leq \alpha.\max\{0,[d(Bz,z)]^2,0\}$$

$$=> d(z,Bz)]^2 \leq \alpha\,(d(z,Bz)]^2$$

$$=> (d(z,Bz)]^2\,(1-\alpha) \leq 0$$

$$=> (d(z,Bz)]^2 = 0 \qquad \because \alpha < 1 => 0 < (1-\alpha)$$

$$=> d(z,Bz) = 0$$

$$=> \ Bz = z$$

$$=> \ Bz = Tz = z \quad , \quad By \quad (1.1.16)$$
....(1.1.19)

By (1.1.18) & (1.1.19)

$Az = Sz = Bz = Tz = z$...(1.1.20)

$\therefore$ We get z is a common fixed point of A, B, S and T.

<u>Uniqueness of Common fixed point :</u>

Let z_1, and z_2 are two common fixed point of A, B, S and T.

Then by using (1.1.20)

$Az_1 = Sz_1 = Bz_1 = Tz_1 = z_1$ ………………………………………………………. (1.1.21)

And also

$Az_2 = Sz_2 = Bz_2 = Tz_2 = z_2$ ………………………………………………………. (1.1.22)

Consider

$$(d(z_1,z_2)]^2 = [d(Az_1,Az_2)]^2 \quad , \quad By \ (1.1.21) \ \& \ (1.1.22)$$

$$\leq \alpha.\max\begin{pmatrix} d(Az_1,Sz_1)\,d(Az_2,Tz_2), \\ d(Az_2,Sz_1)\,d(Az_1,Tz_2), \\ d(Az_1,Sz_1)\,d(Az_1,Tz_2) \end{pmatrix}, \quad By \ (1.1.9)$$

$$= \alpha.\max\begin{pmatrix} d(z_1,z_1)\,d(z_2,z_2), \\ d(z_2,z_1)\,d(z_1,z_2), \\ d(z_1,z_1)\,d(z_1,z_2) \end{pmatrix}, \quad By \ (1.1.21) \ \& \ (1.1.22)$$

$$=> [d(z_1,z_2)]^2 \leq \alpha.\max\{0,[d(z_1,z_2)]^2,0\}$$

$$=> d(z_1,z_2)]^2 \leq \alpha(d(z_1,z_2)]^2$$

$$=> (d(z_1,z_2)]^2 - \alpha(d(z_1,z_2)]^2 \leq 0$$

$$=> [d(z_1,z_2)]^2(1-\alpha) \leq 0$$

$$=> [d(z_1,z_2)]^2 = 0 \qquad \because \quad \alpha < 1$$

$$=> d(z_1,z_2) = 0$$

$$=> z_1 = z_2$$

Hence there exists an unique common fixed point of A, B, S and T.

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