Asymptotic Nature Of Mellin Transform In Electrical Field

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Abstract

Integral transform *techniques widely used for solving linear differential equations in mathematics, especially in Engineering. It is commonly used to solve electrical circuit and systems problems. In this paper we show the relation between the Mellin transform of the derivative of a function is not simple nature as that of Laplace transform. A* transform *table will enable to obtain the solution by a method similar to the method used in Laplace* transforms *theory. Also we focus on some properties of Mellin transform and may be used to solve the Euler-Cauchy differential equation* $\sum_{i=0}^{n} A_i t^i \frac{d^i y(t)}{dt^i} = x(t)$ *which will simplify the solution of such an equation. The applications will illustrate instrumentation and Network analysis problems.*

Keywords

Mellin transform, Differential Properties, Euler-Cauchy differential equation, Instrumentation, Network analysis.

1. Introduction

The first occurrence of the transform is found in a memoir by Riemann in which he used it to study the famous zeta function. However it is the Finnish mathematicians R.H. Mellin (1854-1933) who was the first to give a systematic formulation of the transformation and its inverse. The Mellin transform is a basic tool for analyzing the behaviour of many important functions in mathematics and mathematical physics.

Outlines of the paper-

In first section we discuss some generalities on Mellin transform. In Second section we shows how the relation between the Mellin transform of the derivative of a function is not simple nature as that of Laplace transform. In third section we see the application of Mellin to Euler-Cauchy differential equation and in fourth section applications will illustrate Instrumentation problem and then Network analysis problem. Lastly we conclude.

2. Generalities on Mellin transform

We recall first the definition of Mellin transform.
Let f(t)denotes a complex-valued function of the real, positive variable t. The Mellin transform for *f(t)* will be denoted by $M\,[f;s]$ and defined by

$$M\,[f;s] = F(s) = \int_0^{\infty} f(t)\, t^{s-1}\, dt \qquad (1)$$

Where‘s’ is complex. The basic properties of the Mellin transform follows immediately from those of the Laplace transform since these transforms are intimately connected.
The integral (1) defines the Mellin transform in a vertical strip in the s plane whose boundaries are determined by the analytic structure of f(t) as t tend to 0+ and t tends to +∞.If we suppose that

$$f(t) = \begin{cases} 0\ \ (t^{-\alpha-\varepsilon}) & as\ t \to 0+ \\ 0\ \ (t^{-b+\varepsilon}) & as\ t \to +\infty \end{cases}$$

Where $\varepsilon > 0\ and\ a < b$, then the integral (1) converges absolutely and defines an analytic function in the strip $a < \mathrm{Re}(s) < b$
And the inversion integral formula for (1) follows directly from the corresponding inversion formula for the bilateral Laplace transform. Thus we find the result

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} t^{-s} M[f;s]ds \quad (a < c < b) \qquad (2)$$

Which is valid at all points t ≥ 0 where f(t) is continuous.

3. Nature of Mellin transforms

In common with other integral transforms, the Mellin transform possesses a series of simple translational properties which greatly facilitate the evaluation of transforms of more involved functions.

All these results can be obtained by straight forward manipulation of the definition (1).(See Table-I)

Table-I

SN	Original function	Mellin Transform
1	$f(t), t>0$	$F(s)=\int_0^\infty f(t)\, t^{s-1}\, dt$
2	$f(t)$	$F(s)$
3	$Af_1(t)+Bf_2(t)$	$AF_1(s)+BF_2(s)$
4	$f(at), a>0$	$a^{-s}F(s)$
5	$f\left(\frac{1}{t}\right)$	$F(-s)$
6	$f(t^a),\ a$ real $\neq 0$	$\lvert a\rvert^{-1} F(a^{-1}s)$
7	$(t)^a f(t),\ a>0$	$F(s+a)$

The Mellin transform of derivative of *f(t)* can be found by integration by parts to yield

$$M\,[f'(t);s]=F(s)=\int_0^\infty f'(t)\,t^{s-1}\,dt$$

$$=\left[t^{s-1}f(t)\right]_0^\infty-(s-1)\int_0^\infty f(t)\,t^{(s-1)-1}dt \tag{3}$$

If *f(t)* satisfies (1) ,we have

$$\lim_{t\to 0} t^{s-1}f(t)=0 \text{ for } \quad \operatorname{Re}(s)>a+1$$

$$\lim_{t\to \infty} t^{s-1}f(t)=0 \text{ for } \quad \operatorname{Re}(s)<b+1$$

And hence (3) becomes,

$$M\,[f'(t);s]=-(s-1)F(s-1) \qquad (a<\operatorname{Re}(s-1)<b)$$

$$M\,[f''(t);s]=F(s)=\int_0^\infty f''(t)\,t^{s-1}\,dt$$

$$=\left[t^{s-1}f'(t)\right]_0^\infty-(s-1)\int_0^\infty f'(t)\,t^{s-2}dt \tag{4}$$

And solving same above we get,

$$M\,[f''(t);s]=-(s-1)(s-2)F(s-2)$$
$$(a<\operatorname{Re}(s-2)<b)$$

But the relation between the Mellin transform of the derivative of a function is not simple nature as that of Laplace transform. For e.g.

$$\int_0^\infty \frac{d^n f(t)}{dt^n}t^{s-1}dt=\left[\frac{d^{n-1}f(t)}{dt^{n-1}}t^{s-1}\right]_0^\infty -(s-1)\int_0^\infty \frac{d^{n-1}f(t)}{dt^{n-1}}t^{s-2}dt \tag{5}$$

So that, if we assume*f* to be of such a nature that the square bracket vanishes, we have the relation

$$M\,[f^n(t);s]=-(s-1)F^{n-1}(s-1)$$
$$(a<\operatorname{Re}(s-1)<b) \tag{6}$$

In similar manner we obtain by induction, (Appling this rule until we reach $F^0(s)$)the Mellin transform of n the derivative of f (t) is given by

$$M\,[f^n(t);s]=(-1)^n\frac{\Gamma(s)}{\Gamma(s-1)}F(s-n)$$
$$(a<\operatorname{Re}(s-n)<b) \tag{7}$$

Where,

$$\frac{\Gamma(s)}{\Gamma(s-1)}\equiv(s-n)(s-n+1)......(s-1)$$

This formula gives the Mellin transform of the derivative in terms of the Mellin transform of the function itself.

A similar relation is the Mellin transform of the expression:

$$M\left[t^n\frac{d^n f(t)}{dt^n}\right]=\int_0^\infty t^{s+n-1}\frac{d^n f(t)}{dt^n}dt$$

$$=\left[t^{s+n-1}\frac{d^{n-1}f(t)}{dt^{n-1}}\right]_0^\infty-(s+n-1)\int_0^\infty t^{s+n-1}\frac{d^{n-1}f(t)}{dt^{n-1}}dt \tag{8}$$

Where n=0,1,2……..

where the function is such that the quantity in the bracket again goes to zero. Repeating the process, we get:

$$M\left[t^n\frac{d^n f(t)}{dt^n}\right]=(-1)^n s(s+1)(s+2)$$
$$....(s+n-1)F(s) \tag{9}$$

Where F(s) is the Mellin transform of the function f(t).Other simple relations which can be derived in the same way are:

$$M\left[\left(t\frac{d}{dt}\right)^n f(t)\right]=(-1)^n s^n F(s) \tag{10}$$

And

$$M\left[\left(\frac{d}{dt}t\right)^n f(t)\right]=(-1)^n (s-n)^n F(s) \tag{11}$$

Table-II		
SN	Original function	Mellin Transform
1	t	$-\frac{1}{(s+1)}$
2	t^{-a}	$-\frac{1}{(s-a)}$
3	$\frac{1}{(b-a)}\left[t^{-a}-t^{-b}\right]$	$\frac{1}{(s-a)(s-b)}$
4	e^{-at}	$a^{-s}\,\Gamma(s)$
5	Sin t	$\Gamma(s)\sin\left(\frac{\pi s}{2}\right)$
6	Cos t	$\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$

The convolution or faulting theorem for the Mellin transform is derived in the same way as that of for the Laplace Transform. Let us suppose that F(S) and G(S) be the Mellin transforms of the functions f(t) and g(t)respectively; then the Mellin transform of the product f(t) g(t) is defined to be

$$\int_0^\infty f(t)g(t)t^{s-1}dt=\int_0^\infty g(t)t^{s-1}dt\,\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(\phi)t^{-\phi}d\phi$$

(…using equation (2)) (12)

$$=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(\phi)d\phi\int_0^\infty g(t)t^{s-\phi-1}dt$$

$$=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(\phi)G(s-\phi)d\phi$$

(By equation (1)) (13)

In a similar way the Mellin transform of the product F(s) G(s) is,

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(\phi)G(s)t^{-s}ds$$

$$=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(s)t^{-s}ds\int_0^\infty g(u)u^{s-1}du$$

$$=\int_0^\infty g(u)\frac{du}{u}\,\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(s)\left(\frac{t}{u}\right)^{-s}ds$$

$$=\int_0^\infty f\left(\frac{t}{u}\right)g(u)\frac{du}{u}$$

(..By(1) and(2)) (14)

Table-III		
SN	Original function	Mellin Transform
1	$\left[\left(t\frac{d}{dt}\right)^n f(t)\right]$	$(-1)^n s^n F(s)$
2	$\left[\left(\frac{d}{dt}t\right)^n f(t)\right]$	$(-1)^n (s-n)^n F(s)$
3	$\frac{d^n f(t)}{dt^n}$	$(-1)^n \frac{\lceil s}{\lceil (s-1)} F(s-n)$
4	$t^n \frac{d^n f(t)}{dt^n}$	$(-1)^n (s)_n F(s)$
5	$f(t)g(t)$	$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(\phi)G(s-\phi)d\phi$
6	$\int_0^\infty f\left(\frac{t}{u}\right)g(u)\frac{du}{u}$	$F(\phi)G(s)$

4. Application on Euler-Cauchy differential equation:

Certain types of linear systems give rise to Euler-Cauchy differential equations.

Application of a Mellin transform to this type of equation will yield an algebraic equation.

A Euler's Cauchy differential equation is of the form,

$$\sum_{i=0}^{n}A_i t^i\frac{d^i y(t)}{dt^i}=x(t) \tag{15}$$

Where A_i's are constants.

Apply the Mellin Transform using Table-I& III to obtain the transformed equation.

$$\sum_{i=0}^{n}A_i\, M\left[t^i\frac{d^i y(t)}{dt^i}\right]=M\left[x(t)\right]$$

$$\sum_{i=0}^{n}A_i(-1)^i\,(s)_i Y(s)=X(s)$$

Where i is a positive integer and

$$(s)_n=s(s+1)(s+2).....(s+n-1)$$

$$\sum_{i=0}^{n} A_i (-1)^i s(s+1)(s+2) \ldots .(s+i-1) Y(s) = X(s) \quad (16)$$

$$\therefore Y(s) = \frac{X(s)}{\sum_{i=0}^{n} A_i (-1)^i s(s+1)(s+2) \ldots .(s+i-1)} \quad (17)$$

Using the familiar partial fraction method and a Table of transform pair, the inverse transform of (17) is easily obtained. Hence for linear Mellin transform the sum of the inverse transform of each fraction is equal to the inverse transform of the sum.

5. Application of Mellin transforms in the Instrumentation

If the current which is measured by a meter in a circuit with varying current $i(t) = \frac{1}{t^2} + \frac{1}{t^3}$, and the resistance, $R = \frac{R_0}{t}$ .what is the driving voltage of the network shown in Fig.1 ?

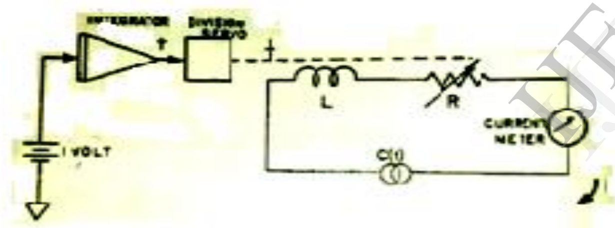

Fig 1

The differential equation for Fig.1 is:

$$e(t) = L\frac{di(t)}{dt} + Ri(t) = L\frac{di(t)}{dt} + \frac{R_0}{t} i(t) \quad \text{Or}$$

$$te(t) = Lt\frac{di(t)}{dt} + R_0 i(t) \quad (18)$$

Taking Mellin transform, we get

$$M\{te(t)\} = L\,M\{t\frac{di(t)}{dt}\} + R_0\,M\{i(t)\}$$

$$E(s+1) = -L\,SI(S) + R_0 I(S)$$

$$= I(S)(R_0 - L\,S)$$

(using Table -I) (19)

,where $i(0) = \frac{di(0)}{dt} = 0$

If the current meter reads, $i(t) = \frac{1}{t^2} + \frac{1}{t^3}$ then the equation by transforming (Table-II)we get,

$$I(S) = \frac{-2s+5}{(s-2)(s-3)} \quad (20)$$

Therefore equation (19) becomes,

$$\therefore E(s+1) = \frac{(-2s+5)L\left[\frac{R_0}{L} - s\right]}{(s-2)(s-3)}$$

$$= \frac{L(3-a)}{s-2} - \frac{L(a-s)}{s-3} \quad \because \frac{R_0}{L} = a$$ (by using partial fraction method) (21)

Taking Inverse Mellin transform of both sides by using Table II, we get

$$t\,e(t) = (3-a)Lt^{-2} - (2-a)Lt^{-3} \quad \text{Or}$$

$$e(t) = \frac{L}{t^3}\left[(3-a) - \frac{(2-a)}{t}\right] \quad (22)$$

which is the voltage necessary to provide the given current.

6. Application of Mellin transforms in Network analysis

When the switch S is closed, current i(t) is measured by a meter in a circuit is given by $i(t) = \frac{1}{t} + \frac{1}{t^2}$, and the Capacitor $\frac{1}{c^2} = \frac{c_o}{t^2}$,the resistance, $R = \frac{R_0}{t}$ find the driving voltage $e(t)$ for the network shown in Fig.2 ?

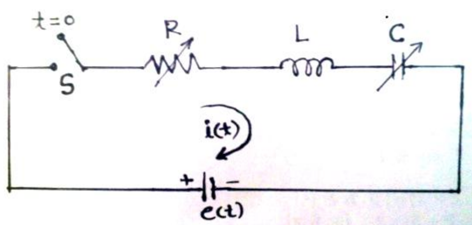

Fig.2

The differential equation for Fig.2 is:

$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{c}\int i(t)dt \quad (23)$$

$$e(t) = L\frac{di(t)}{dt} + \frac{R_0}{t} i(t) + \frac{c_o}{t^2}\int i(t)dt \quad \text{OR}$$

$$te(t) = L\,t\,\frac{di(t)}{dt} + R_0 i(t) + c_0 \int \frac{i(t)}{t}\,dt \qquad (24)$$

Taking Mellin transform (Table-I & II), we get,

$$E(s+1) = -L\,SI(S) + R_0 I(S) - \frac{C_0}{s} I(S)$$

$$= \left[-L\,S + R_0 - \frac{C_0}{s}\right] I(S)$$

$$= \left[-S + a - \frac{b}{s}\right] I(S) \qquad (25)$$

$$\because \text{a} = \frac{R_0}{L} \quad b = \frac{C_0}{L}$$

Where $i(0) = \frac{di(0)}{dt} = 0$

If the current meter reads, $i(t) = \frac{1}{t} + \frac{1}{t^2}$

Equation by transforming we get,

$$I(S) = \frac{-2s+3}{(s-1)(s-2)} \qquad (26)$$

Therefore equation (25) becomes,

$$\therefore E(s+1) = \frac{(-2s+3)L\left[-S + a - \frac{b}{s}\right]}{(s-1)(s-2)}$$

(by using partial fraction method)

$$= \frac{-L(\text{a} - b - 1)}{s-1} - \frac{L\left(a - b/2 - 2\right)}{s-2} \qquad (27)$$

Taking Inverse Mellin transform of both sides by using Table II, we get

$$t\,e(t) = L(\text{a} - b - 1)t^{-1} + L\left(a - b/2 - 2\right)t^{-2}$$

$$e(t) = L(\text{a} - b - 1)t^{-2} + L\left(a - b/2 - 2\right)t^{-3}$$

Or

$$e(t) = \frac{L}{t^2}\left[\frac{\left(a - b/2 - 2\right)}{t} + (\text{a} - b - 1)\right] \qquad (28)$$

which is the voltage necessary to provide the given current.

Conclusion

The use of the Laplace integral transform for some of the random variables is mostly used and explained in every advanced Engineering and Science field, now brief theory of Mellin integral transform for electrical Engineering is given in this paper .It seems for any statisticians, mathematicians and engineers will also take interest in developing Mellin transform.

Here the paper presented some background on Mellin transform theory and motivated to compute the Euler-Cauchy differential equation $\sum_{i=0}^{n} A_i t^i \frac{d^i y(t)}{dt^i} = x(t)$ and the application of Mellin transform in different areas of Electrical Engineering. It is a very effective mathematical tool to simplify very complex problems in the area of Instrumentation and Network analysis.

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