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Boundary Element Methods for Thermal Problems - Review

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Abstract

Boundary element methods (BEM) is attractive in solving several engineering problems.In recent years BEM has been developed as a competitive method with the other numerical methods like Finite Difference Method (FDM) and Finite Element Method (FEM). This research paper explores the history and developments of the boundary element method BEM form early 1900's to present day.The BEM has been applied to a variety of heat transfer problems in the last thirty years. The problem of solving heat transfer problems is of great interest to a wide-range of engineers and scientists. This paper presents how BEM is successfully applied to heat transfer problems i.e., steady and unsteady heat conduction problems and thermal radiation problems. In this paper an effort is made to show the benefits of BEM compared to other numerical methods and itsapplication in solving the heat conduction and radiation problems is discussed.

KEY WORDS: Boundary Element Method, Heat conduction, Thermal radiation, Numerical methods

1. Introduction

The science of thermodynamics deals with the amount of heat transfer as a system undergoes a process from one equilibrium state to another, and makes no reference to how long the process will take. But in engineering, we are often interested in the rate of heat transfer, which is the topic of the science of heat transfer. The review of the fundamental concepts of thermo- dynamics that form the framework for heat transfer is explained. The relation of heat to other forms of energy and review the first law of thermo-dynamics is necessary. The three basic mechanisms of heat transfer, which are conduction, convection, and radiation. Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent, less energetic ones as result of interactions between the particles. Convection is the mode of heat transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion. Radiation is the energy emitted by matter in the form of electromagnetic waves (or photons) as a result of the changes in the electronic configurations of the atoms or molecules.

If a researcher or engineer wants to solve simple heat transfer problems involving simple geometries like sphere, cylinder etc. with simple boundary conditions, it can be solved by analytical methods. But many problems in real situation involve non-conical geometries with complex boundary conditions or variable properties and cannot be solved analytically. In such cases, sufficiently accurate approximate solutions may be obtained by computers using numerical methods.

Analytical solution methods depends upon the solution of governing differential equations together with boundary conditions. They result is solution function for temperature at every point in the medium. On the other side, Numerical methods are based on replacing the differential equation by a set of algebraic equations for the unknown temperatures at selected points in the medium, and the simultaneous solution of these equations results in value of temperature at those discrete points.

There are several ways of obtaining the numerical formulation of a heat transfer problem such as FDM, FEM, BEM and Finite Volume Method (FVM). Each method has its own advantages and limitations.

Modern computational techniques facilitate solving problems with imposed boundary conditions using different numerical methods. Numerical analysis of heat transfer has been independently though not exclusively, developed in three main streams: the finite differences method the finite element method and the boundary element method. FEM and FDM fits well for heat conduction problems. Using of FEM and BEM will be advantage for heat radiation problems. FVM and FDM fits well for heat convection problems.

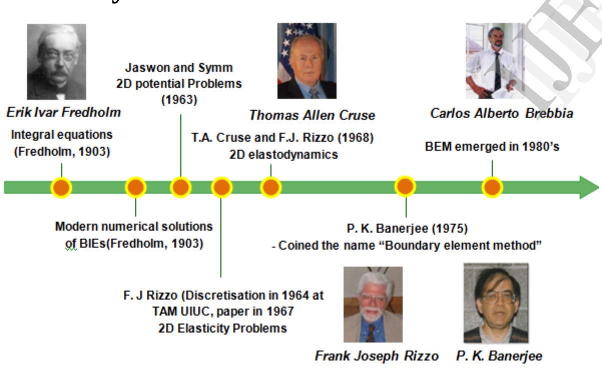
The FDM is based on the differential equation of the heat conduction, which is transformed into a numerical one. The temperature values will be calculated in the nodes of the network. Using this method convergence and stability problem can appear.

In FEM equations are obtained from the differential equation using variational calculus. In first case the temperature values will be calculated on the finite elements. Then, based on these partial solutions, the solution for the entire element will be determinated. In case of BEM the boundary is discretized into elements and internal point position can be freely defined.

The most popular are mesh methods, such as the FEM and the FDM. Although this methods are well established and commonly applied to transfer heat analysis, in many problems, mesh generation can be very laborious and constitutes the most expensive and difficult part of numerical simulations. Moreover, in objects of complex geometries,generated meshes can be distorted, what contributes to increase of computational error (Li, 2011) [1].The BEM has been applied to a variety of heat transfer problems in the last thirty years. The BEM is often referred to as the Boundary Integral Method (BIM) or Boundary Integral Equation Method (BIEM).

The BEM requires discretisation of the surface rather than discretising the entire continuum by FEM. The BEM method is used for more complex mathematical calculations. The computer code written for BEM is easier and simpler to use as compared to FEM. BEM is emerging as an alternative to the FEM from recent years, mainly for the problems where domain extent to infinity. BEM can be competitive to FEM or FDM for solving thermal engineering problems.

2. History of BEM:



The available statistical data shows that the amount of annual published literature described by BEM saw an exponential development until late 1990's and exceeded 700 research articles in the successive years. Famous mathematicians and scientists worked and presented their approaches in BEM. The contributions and the efforts of mathematicians like Laplace, Green, Fredholm, Fourier, Kellogg and Betti could be traced in the theoretical and mathematical foundation of BIE in the early20thcentury. While the works of Somigliana (1885/86)[2,3], Fredholm (1903)[2,4], Kupradze (1965)[2] and many others can be seen as precursors to the development of Boundary Element Methods.The publications of Jaswon (1963) and Symm (1963) [2,5-7] can be arguably be considered as their origin. In these works, Jaswon and Symm have developed direct BIEM for potential problems using Green's third identity. Based on this approach, Rizzo elastostatic problems using Somigliana's identity has presented a formula for transient elastodynamics employing the Laplace transform (Cruse and Rizzo (1968), Cruse (1968)) [8-11]. Later, their work has attracted and inspired researchers to investigate the potential of the new boundary integral approach. Considered as a major breakthrough in BEM, their work is then adopted by many researchers and saw rapid progress in the development of boundary integral equation approach.

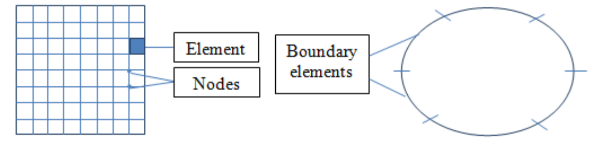
Important contributors to the numerical implementation of the method are due to Lachat and Watson (1976) [12], who introduced sub-regions to handle large-scale problems and described algorithms for the computation of the singular and quasi-singular integrals that appear in the boundary integral equations. An algorithm for the computation of Cauchy principal value integrals resulting from the strong singularity of the traction fundamental solution has been formulated by Guiggiani and Gigante (1990) [13]. The term 'Boundary Element Method' was coined in the three publications (Banerjee and Butterfield (1977)[110], Brebbia& Dominguez (1977) [14], and in the following year, the first book on the method appeared (Brebbia (1978)) [15].

In 1978, Brebbia published the first textbook on BEM 'The Boundary Element Method for Engineers'. The book contained a series of computer codes developed by Dominguez. In the same year, Brebbia organized the first conference dedicated to the BEM, the First International Conference on Boundary Element Methods, at the University of Southampton. This conference series has become an annual event and is nowadays organized by the Wessex Institute of Technology. The most recent in the series as of this writing is the 26th conference in 2004. In 1984 Brebbia founded the Journal 'Engineering Analysis-Innovations in Computational Techniques' [16,17]. It initially published papers involving boundary element as well as other numerical methods. In 1989, the journal was renamed to 'Engineering Analysis with Boundary Elements' and became a journal dedicated to the boundary element method.

Thanks to early development prepared by numerous researchers who were directly involved in constructing the blueprint of modern BEM which was then well received by the scientific community, That now a days the BEM has been adopted by many researchers covering countless specialized areas such as acoustics, contact mechanics, dynamics analysis, solids and structures, soil-structure inter-actions, nonlinear fluid dynamics, heat transfer and quantum mechanics. With the solid foundation and rich heritage, BEM emerged as a powerful method and thus become a strong alternative to the FEM and FDM.

3. The Features of BEM:

1. In BEM only boundary needs to be discretised, which will lead to easy data preparation and less computing requirements. Problem size has been reduced because of the reduction in size of the mesh.



Finite element method *Boundary element method*

2. Solution on the boundary is calculated first and then the solution at domain points are found as a separate step. There are many problems where the details of interest occur on the boundary or localised to a particular part of the domain and hence an entire domain solution is not required.
3. Reactions on the boundary are typically more accurate than the dependent variables.
4. Non symmetric matrices are generated, the matrices are generally of different sizes due to the differences in size of the domain mesh compared to the surface mesh. There are problems where either method can give rise to the smaller system and quickest solution - it depends partly on the volume to surface ratio. For problems involving infinite or semi-infinite domains, BEM is to be favoured.
5. Integrals are more difficult to evaluate, and some contain integrands that become singular. BEM integrals are far harder to evaluate. Also the integrals that are the most difficult (those containing singular integrands) have a significant effect on the accuracy of the solution, so these integrals need to be evaluated accurately.
6. In BEM only boundary conditions are being approximated; but in other numerical methods (FEM/FDM) differential equation is being approximated. The use of the Green-Gauss theorem and a fundamental solution in the formulation means that the BEM involves no approximations of the differential Equation in the domain - only in its approximations of the boundary conditions.
7. BEM cannot handle all linear problems. A fundamental solution must be found before the BEM can be applied. There are many linear problems like nonhomogeneous equation for which fundamental solutions are not known. There are certain areas in which the BEM is clearly superior, but it can be rather restrictive in its applicability.
8. Implementation of BEM is more difficult. The need to evaluate integrals involving singular integrands makes the BEM at least an order of magnitude more difficult to implement than a corresponding finite element procedure.

4.0 Application of BEM for Heat transfer problems:

4.1 Heat Conduction Problems:

The heat transfer in solids, with the changes of temperature in time on physical boundaries of analysed objects, occur in many engineering mechanisms (engines, compressors), heating and cooling systems and hydraulic networks (Zhang et al., 2009; Lu and Viljanen, 2006) [18-20]. The analysis of basic mechanism of heat transfer in solids, that is heat conduction problem, is significant for process of designing and optimization mechanical systems and devices. Accordingly, the heat conduction equations with conditions of variable temperature or heat flux on boundaries become an important instrument for mathematical description of many engineering, geothermal and biological problems. As a result, there is a need to develop effective computational methods and tools for solving transient heat conduction problem (Mansur et al., 2009; Yang and Gao, 2010) [20,21].

Conduction of heat means transfer of heat energy within the body due to the temperature gradient. Heat spontaneously flows from a body having higher temperature to lower temperature. But in absence of external driving fluxes it approaches to thermal equilibrium.

There are two types of conduction such as:

1) Steady state conduction and
2) Transient or Unsteady state conduction

Steady state conduction:

Steady state conduction is a form of conduction where the temperature differences deriving by the conduction remains constant and it is independent of time. Steady state conduction happens when the temperature difference driving the conduction are constant after an equilibration time, the spatial distribution of temperatures (i.e. temperature field) in the conducting object does not change further. Thus, all partial derivatives of temperature with respect to space may either be zero or have nonzero values, but all derivatives of temperature at any point with respect to time are uniformly zero. In steady state conduction, the amount of heat entering any region of an object is equal to amount of heat coming out. The steady state heat conduction problem is well known. The analysis will be difficult for the problems like, where heat transfer takes place through a complicated domain. Different numerical methods are proposed for these types of problems, e.g. finite difference method, finite volume method and finite element method.

Transient or unsteady state conduction:

In transient heat conduction temperatures are varying with time. In unsteady-state heat conduction analysis the temperature is governed by an elliptic partial differential equation. These equations are not difficult to solve. Numerical methods must be used to solve if the boundary conditions are nonlinear or if the system domain are irregular in shape; but the problems cannot be solved exactly.
Two groups of method are applied to obtain transient heat conduction problem solution: analytical and numerical. In the literature, many analytical methods have been proposed, inter alia based on orthogonal and quasi-orthogonal expansion technique, Laplace transform method, Green's function approach or finite integral transform technique, but they are feasible only for problems with simple geometries (Singh et al., 2008) [22]. In spite of development of analytical techniques, this methods still cannot be employed for solving most practical heat transfer problems, such as heat conduction in anisotropic materials, objects of complex geometries or complex boundary conditions (Rantala, 2005; Johansson and Lesnic, 2009) [23-25]. Hence, for last few decades, the numerical methods have been strongly developed, as more universal computational tool.

Monte et al. (2012) [26] presented very accurate analytical solutions modeling transient heat conduction processes in 2D Cartesian finite bodies, such as rectangle and two layer objects, for small values of the time. In their research work, the geometry criterion was provided that permit to use 1D semi-infinite solutions for solving 2D finite single- and multi-layer transient heat conduction problems. Yumrutas (Yumrutas et al., 2005) [27] developed new method based on Complex Finite Fourier Transform (CFFT) technique for calculation of heat flux, through multilayer walls and flat roofs, and the temperature on the inner surface. The periodic boundary conditions were assumed, that is hourly changeable values of external air temperature and solar radiation. Lu et al. (Lu et al., 2006; Lu and Viljanen, 2006) [18,19] adopted the Laplace transform to solve the multidimensional heat conduction in composite circular cylinder and multilayer sphere, with time-dependent temperature changes on boundary, which were approximated as Fourier series. Singh et al. (2008) [22] applied separation of variables method to obtain analytical solution, in the form of transient temperature distribution, to the 2D transient heat conduction problem in polar coordinates with multiple layers in the radial direction. Rantala (2005) [23] proposed a new semi-analytical method for the calculation of temperature distribution along the fill layer underneath a slab on-ground structure subjected to periodic external and internal temperature.

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The most used mesh methods are FEM and FDM. Although this methods are well established and commonly applied to transfer heat analysis, in many problems, mesh generation can be very laborious and constitutes the most expensive and difficult part of numerical simulations. Moreover, in objects of complex geometries, generated meshes can be warped, that contributes to increase of computational error (Li, 2011) [1].

The numerical solution of heat conduction problems are classified into two categories, such as:

1) Whole domain approach
2) Boundary approach

The whole domain approach consists of the popular FDM and FEM. These methods discretise the whole domain into a elements. Those elements on the boundaries are thus analysed together with the interior points to solve for the temperature. There are major differences, however, in the ways the finite difference and finite element equations are derived in the solution. In the FDM, a local energy balance is invoked at the nodal points in order to derive a set of algebraic equations; whereas in the FEM, the algebraic equations are derived on the basis of satisfaction of the governing partial equation in a global sense. It is more difficult to derive the finite element equations, but the FEM are more convenient to use in the solution of problems in irregular domains. Such an advantage is diminishing recently, however, because of grid generation techniques.

The drawback of mesh generation is overcome in the meshless methods, that use a set of scattered nodal points in considered object (no connectivity among nodes), instead of meshes (Cheng and Liew, 2012; Ochiai et al., 2006) [20,28,29]. Some of this methods have been recently applied to transient heat conduction analysis in 2D objects, like Meshless Element Free Galerkin (EFGM) method (Zhang et al., 2009) [30], Meshless Local Petrov-Galerkin (MLPG) method (Li et al., 2011) [1], Method of Fundamental Solutions [MFS] (Johansson and Lesnic, 2008, 2009) [24,25], meshless local Radial Basis Function- Differential Quadrature (RBF-DQ) method (Soleimani et al., 2010) [31], and in 3D objects, like Meshless Reproducing Kernel Particle Method (RKPM) (Cheng and Liew,2012) [20]. These methods are more time-consuming than mesh methods, such as FEM because of the larger dimensions of generated matrices (Zhang et al., 2009) [30].

The alternative solution for mesh and mesh free methods is BEM. Compared to FDM and FEM, the great advantage of BEM is the possibility of determination of the solution (both the function and the derivative of this function) at any point of the domain without necessity of construction of grids. The discretization is performed only over the boundary, not over the whole analyzed domain hence the size of system of equations, that need to be solved, is reduced by one. In BEM, the fully populated coefficient matrices are generated, what is the opposite of banded and symmetric matrices in FEM. However, the small dimensions of BEM matrices counterbalance this limitation (Katsikadelis, 2002; Majchrzak, 2001; Pozrikidis, 2000) [32-34]. Application of the BEM requires the knowledge of fundamental solution of the governing differential operator, but at the same time, the use of fundamental solution stabilize the numerical commutations (Ochiai et al., 2006) [29].

The boundary approach of numerical solution is primarily represented by the BIEM, which are commonly known as the BEM. In these methods, the governing partial differential equations are used in conjunctions with the boundary conditions to derive an integral equation, which consists of contour and domain integrals. Great simplicity arises in situations where there are no heat sources and sinks in the domain; then the domain integral vanishes, and only contour integrals remain. It follows that only the system boundaries need to be discretised. Dealing solely with the boundary, the boundary approach is able to simplify element generation; it also reduces the dimension of the discretisation by one. The number of equations solved simultaneously in the boundary approach is less than the whole domain approach. However, the coefficient matrices generated in the boundary approach are unsymmetric and the matrix elements are nearly fully populated. There have been a number of studies devoted the comparison of the boundary approach with the whole domain approach.

The BEM is successfully applied to steady and unsteady heat conduction problems. As opposed to steady problem, in mathematical description of transient heat conduction, the domain integrals occur. In order to keep the boundary character of the method, many different techniques have been developed, but the most popular are: method using the Laplace transformation to eliminate the time derivative, the dual reciprocity method, and the convolution scheme (employing time-dependent fundamental solutions).

Erhart (Erhart et al., 2006) [35] implemented the Laplace transformation for solution of transient heat transfer in multi-region objects. As a result the time-independent boundary integral equation was produced, solved further with a steady BEM approach. The last step was numerical inversion of the solution, done with the use of Stehfest method. The derived algorithm was applied to heat conduction in a bar, laminar airfoil with three cooling passages and non-symmetric airfoil. The results were compared with those obtained with finite volume method (FVM).

Sutradhar and Paulino (2004) [36] also used the Laplace transformation, both with Galerkin approximation, for analysis of the nonhomogenous transient heat conduction problem in functionally graded materials FGM of variable thermal conductivity and specific heat. The three kinds of material variation, that is quadratic, exponential and trigonometric, were assumed for verifying the accuracy of presented method. The practical example for thefunctionally graded rotor problem was carried out.

Another approach is Fourier transform, applied by Simoes (Simoes et al., 2012) [37] and Godinho (Godinho et al., 2004) [38], consist in three general steps: converting analyzed domain into frequency domain, solving the heat conduction problem with BEM and obtaining the final solution in time domain with the use of inverse Fourier transform. Simoes tested method in 2D object with unit initial temperatures and with non-constant temperature distribution in domain. Godinhoanalyzed transient heat conduction around a cylindrical irregular inclusion of infinite length, inserted in a homogeneous elastic medium and subjected to heat point sources placed at some point in the host medium.

Mohammadia (Mohammadia et al., 2010) [39] solved 2D nonlinear transient heat conduction problems with non-uniform and nonlinear heat sources, with the new BEM approach, using timedependent fundamental solutions. In this method temperature is computed on the boundary and in internal points at every time step, and the results constitute the initial values for the next time step. However, for 3D and large problems, the storage of coefficient matrices for every time step can be problematic (Erhart et al., 2006) [35].

Tanaka et al. (2006) [40] applied dual reciprocity boundary element method (DRBEM) for analysis of 3D transient heat conduction problem in nonlinear temperature-dependent materials. In proposed method, domain integral is transformed into boundary integrals with the use of radial basis functions. To entertain the material nonlinearity, the iterative solution procedure was employed. Białecki et al. (2002) [41] proposed the DRBEM without matrix inversion for linear and non-linear transient heat conduction problem, that reduce the time of computations. The method was applied to solve heat transfer problem in a turbine rotor blade. Ochiai (Ochiai et al., 2006; Ochiai and Kitayama, 2009) [28,29] developed the triple-reciprocity BEM to solve 2D and 3D transient heat conduction problems. One of the recent methods is radial integration boundary element method RIBEM applied to transient heat conduction problem by Yang and Gao (2010) [21], which can be employed to analysis of functionally graded material problems.

4.2 Application of BEM for Thermal radiation problems:

Thermal radiation is transferred by electromagnetic waves or photons, which may travel over a long distance without interacting with a medium. Thermal radiative energy consists of electromagnetic waves or photons. All electromagnetic waves or photons propagate through any medium at a speed of light. Thermal radiation is a very complex phenomenon and although the governing equations are known but they are difficult to solve. This difficulty is due to radiation intensity as a function of position, direction, wavelength and temperature. In some cases, these dependencies are not straightforward. The heat waves have wave length of 0.1 to 100 microns and most of the bodies emit radiation over a range of wavelength.

Heat transfer by thermal radiation is used for different applications combustion, propulsion, cryogenic devices, boilers, industrial furnaces, solar energy equipments, satellites, glass manufacture and energy conservation. These applications cannot support thermal conduction or convection, since they need a vacuum environment. The importance of the thermal radiation has motivated the need to develop analytical and numerical techniques.

Conductive and convective heat transfer rates are linearly proportional to temperature differences but radiative heat transfer rates are proportional to differences in temperature to the fourth power. Thermal radiation being part of electromagnetic spectrum travels with the speed of light. The speed of light is so large compared to local time scales and length scales that the transient term from the radiative transfer equation is neglected i.e. radiation is assumed to be an instantaneous (steady-state) process. The transient term of the radiative transfer equation (RTE) can be neglected i.e. steady-state RTE assumption does not lead to significant errors, since the temporal variations of the observables are in the range of 10-9 to 10-15.

The thermal radiation equation is expressed in terms of the radiation intensity and is an integro-differential equation. The governing equation for thermal radiation is expressed in the following integro-differential form.Radiative transfer equation is an integro-differential equation involving several independent variables such as wavelength of radiation, three space coordinates(x, y and z), Two coordinates describing the direction of travel, polar angle and azimuthal, angle (r, θ and φ) and Time. Therefore its analytical analysis is very challenging.

The various numerical methods have been used to solve the radiative transfer equation are the Monte Carlo method, the integral equation solution, the FVM, the radiation element method (REM), discrete transfer method (DTM) and the discrete ordinates method (DOM), flux method, BEM. All these methods have their advantage and limitations.

To solve thermal radiation problems energy conversation equation is used along with integro-differential equations. The integro-differential equations and energy conversation equation are solved simultaneously for temperature and the heat flux both on the enclosure surfaces and within the radiation participating medium. It is important to consider the mathematical difficulties involved in solving these equations. The exact solution of the equation of thermal radiation may require integration with respect to time, position, wavelength and solid angle. These complexities can be reduced by considering certain assumptions to simplify the problem. The assumptions considered are isotropic scattering, diffuse gray enclosure walls and gray medium and diffuse gray enclosure walls etc. It is difficult to implement the governing equations in the conventional forms by using numerical methods. To simplify the numerical calculations various exact expressions were derived for thermal radiation analysis in a gray, emitting, absorbing and isotropic scattering medium bounded by a gray and diffusely emitting and reflecting enclosure. Tan.s formulation is popular [42] of all the formulation.

Sparrow (1960) [42-45] developed a variational method to solve the surface heat exchange problems. Monte Carlo method is the first numerical method developed by Howell and Perlmutter (1964) [44-46] to solve the thermal radiation problems. The Zonal method [44,45], the Monte Carlo method, the finite volume method, the discrete ordinate method, the spherical harmonics method, etc. are traditionally used numerical methods in analysing thermal radiation problem. The finite element method (FEM) and boundary element method (BEM) are not along with these methods. FEM has not shown significant growth in solving thermal radiation problem for different applications. FEM is used in Applied Mechanics, Electromagentics and Conduction heat transfer. The use of BEM in thermal radiation problems is rarely reported.

There are several numerical methods used to solve thermal radiation problems. These methods include the Zonal method [45], the Monte Carlo method [44,45], the finite volume method, the discrete ordinate method [the spherical harmonics method etc. [47-52]. It is interesting to notice that the commonly numerical methods for solving engineering problems are FEM. The boundary element is not commonly used as compared to other methods. The use of FEM is new for the field of thermal radiation problems as compared to other fields such as applied mechanics and electromagnetic and the application of the BEM in thermal radiation analysis is very less. From last few decades BEM is emerging as alternative to FEM for engineering problems. The important feature of BEM is that it requires discretization of the boundary not the complete continuum, as required by other numerical methods. Therefore, computer code written in BEM is easier to use than the other numerical methods [53,54].

The idea behind BEM consists in transformation the original boundary-value in to an equivalent integral equation. The latter is then discretised and the resulting set of algebraic equations is solved yielding the desired value of the unknown function and its derivative. Thermal radiation is a phenomenon ruled by an integral equation. FEM and the FDM are developed to solve differential equations. Therefore FEM and FDM required lot of modifications to fit for thermal radiation problems. BEM, as a technique of discretisation of integral equations, seems to be the proper tool to handle suitable for thermal radiation problems. Bialecki (1993) [51,53,54] showed that Kernels functions arising in the integral equations of thermal radiation have asymptotic singular behaviour. Both the Kernels depend on the distance between the observation and current points in the power of -2. All these features cause that the idea of using BEM to solve thermal radiation problems arise in a natural way.

When compared with other numerical methods approaches, BEM offers substantial computing time economy due to the reduction of the integration dimension and lack of volume integrals. Another distinguished feature of BEM is its sound mathematical background. Earlier studies of Weceland Bialeckishow that BEM is a robust tool of simulating radiative heat transfer.Thermal radiation phenomenon governed by integro-differential equation. The thought of using BEM for solving thermal radiation problems naturally arises because of the nature of thermal radiation. BEM and thermal radiation analysis should be a good combination based on few publications.

R.A Bialecki [53,54] published book on solving thermal radiation problems using the boundary element methods. The author has shown numerical examples in the book for illustration of the advantage of BEM in solving thermal radiation problems especially in two dimensional cases. In the book, the governing equations are solved directly using the direct boundary element method, where the full potential and the elegance of BEM as a basis method are not fully used. The scattering effect is assumed to be negligible, which makes the formulations in the book of limited value when handling the general cases of thermal radiation problems. The scattering effect is not negligible in the general cases of thermal radiation. The author has highlighted that including scattering effect in the thermal radiation problems leads to huge mathematical and numerical difficulties.

According to Tan's formulation [55,56], when participating medium is involved in radiative field analysis, domain integrals are present in addition to surface integrals. Here, full form of governing equations has to be considered for any numerical calculations and numerical means, domain integrals are calculated by domain discretisation. To solve general case of thermal radiation problems by using BEM, the discretisation is no longer restricted only to the boundary. The domain has to be discretised. Modified boundary element method is alternative for Tan's formulation to describe the thermal radiation problems in a gray, absorbing, emitting and isotropic scattering medium bounded by a gray and diffusely emitting and reflecting enclosure. The modified boundary element method was found to be efficient and precise in solving general thermal radiation problems.

5. Conclusion

This research paper explores the history, developments and application of the BEM to a variety of heat transfer problems. The problem of solving heat transfer problems is of great interest to a wide-range of engineers and scientists. TheBEM can be successfully applied to heat transfer problems i.e., steady and unsteady heat conduction problems and thermal radiation problems. In this paper an effort is made to show the benefits of BEM compared to other methods and its application in solving the heat conduction and radiation problems is discussed. Several BEM approaches have been developed to solve Heat conduction and radiation problems. Each method has its own advantage and disadvantages in solving Heat transfer problems. New approaches in BEM to be developed and implemented to improve the efficiency of the solution meanwhile keep the classiness of the boundary element method.

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