Brief History And Paradoxes On Primitive Notion Of Set

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Abstract

Here we present a brief early history of set theory. We shall find that in due course several paradoxes were discovered and people began to worry as if the set theory was inconsistent, no mathematical proof could be fully relied . We shall discuss here only some paradoxes on the primitive notion of sets due to Cantor and Russell. Attempts were made to modify the concept of set so that the paradoxes may not built. Such an attempt was made in early days by Ackermann. We shall present here new interpretation of Cantor's words so that the paradoxes may not arise in set Theory.

1. Introduction

Set Theory provides a language which is simple from out side but its internal structure is very rich and its methods are sufficiently universal to formalize all mathematical concepts.

Set theory together with predicate calculus constitutes a strong foundation of Mathematics. George Cantor may be said the father of set theory but later on his set theory gave birth to many paradoxes (contradictions). It would be our endeavour to consider only some paradoxes on the primitive concept of set.

2. Origin of Set Theory

The idea of infinity has been a subject of interest and discussion from the very early period of the human civilization. Set theory had been considered as a science of the infinite, consequently the origin of Set theory is as old as the infinity (around 450 BC) itself.

3. Historical development :

The seeds of set theory grew gradually and in 1847 Bolzano a great Philosopher and mathematician described a set as "an embodiment of the idea or concept which we conceive when we regard the arrangement of its parts as a matter of indifference."

He defended the concept of infinite sets. He showed with the help of examples that the elements of an infinite set could be put in one-one correspondence with one of its proper subsets. But due to some paradoxes that arose, he could not develop set theory, though some of his ideas appear in Cantor's work.

The history of set theory reveals that it is the creation of one man, George Cantor. Cantor was the person who put set theory on a proper mathematical base. Between 1867 and 1870 Cantor worked on number theory but in fact his works were of such a quality that brought a change in the entire world of Mathematics.

The year 1872 was of much importance when Cantor met Richard Dedekind in Switzerland. Dedekind had possessed a deep logical sense of thinking and influenced Cantor to develop his ideas.

The real date of birth of set theory may be taken as December 7, 1873, the date on which Cantor informed Dedekind by letter about his discovery.

4. Some Paradoxes on the primitive notion of set :

In 1899 a paradox was discovered by cantor himself which arises from the set of all sets. In 1901 while working on his Principles of mathematics (1903) Russell discovered his paradox and this paradox is the most famous of set theoretical paradoxes. The Paradox arises within Naiv set theory created by Cantor by considering the set of all sets that are not members of themselves. In the set S = { x/ x is not a member of x }. Is S an element of S ? Both assumptions that S is a member of S and S is not a member of S lead to contradiction.

(Though the same paradox had been discovered one year before by Emst Zermelo but he could not publish).

To prevent the paradox, Russell devised theory of types. Russell's solution does not succeed in avoiding the contradictions, mathematicians decided that the solution should be more intuitive for the foundation of mathematics.

According to Cantor himself, the paradoxes arose due to misusing his definitions.

It is remarkable that Ackermann worked very well set theory by putting a different interpretation on Cantor's words.

5. New interpretation of Cantor's words :

Let us first considered only objects of thought "in constructing pure set theory leaving out primitive individuals at first. The objects themselves are regarded as indefinite but only to the extent that classes or sets to be formed can contain them as elements. These objects are of two categories'-classes and sets. the sets are classes of objects collected with the help of properties (predicates) P(x). These predicates are to be formed from the basic predicates equality and membership with the usual logical connectives. One more basis predicate. T(x) : "x is a set" is also assumed. The only restriction to be observed is that in constructing a set x, the predicate T(x) should not be used.

a. Let P (x) be a predicate of such a kind that everything x possessing the predicate (i.e., property) P (x) is a set. There will then exist first a class of objects x with the property P(x), so that “x is an element of this class” and P(x) will be regarded as equivalent terms.

b. Classes with the same elements will be identical.

c. Not every class of sets is a set. Cantor requires for this that a set contains only definite and distinct objects. As we have to deal with only classes of sets, in order that a class is a set, the sets forming the class must be clearly defined so that it may be definite, what belongs to the class and what not. The concept of a set is thus completely open. Cantor’s definition is so intended that a class can be examined in each case as to whether it represents a set or not. The definition is not so intended that it can be determined for all classes at once whether they are sets or not. We can then only take a class to be a set which is defined with predicates P(x) as considered above. But the condition being that the defining predicate P(x) should not have T(x) as one of its constituents, though, P(x) may be constructed in terms of other already constructed sets.

d. The assumption is in general theories of sets that “the elements of sets are themselves sets” becomes superfluous with special sets - it follows from the general definition of the set. Further, if a class is contained in a set t (i.e., if all elements of the class are elements of a set t) we can assume the set character of the class; as the inclusion in a set t itself means that all elements of the class, elements of elements of the class and so on, are sets. So, without reference to the general definition of a set, we can assume that every class which is contained in a set is itself a set. Precaution should be taken when the property P(x) contains other parameters in addition to x - in that case it should be emphasized that parameters are set variables.

These requirements are such that the paradoxes cannot be built in set theory. For example, consider the paradox of the ‘set of all sets’. Though for P(x) = ‘x is a set’ there is a class of all sets (according to (a), this class is not a set, since the requirements in (c) are not satisfied.

Next, let us consider Russell’s paradox regarding the ‘set of all sets not containing themselves as elements. The class corresponding to ‘x is a set and x is not an element of x' does not fulfil condition (c) . In accordance with (c), if we leave out the offensive part T(x)-x is a set-then corresponding to 'x is not an element of x' there is no class, as there is no certainty that x is a set.

If we take P(x) = 'x is different from x', then requirement (a) is satisfied, as every x with this property is a set (as there is no x of this kind). Further according to (c) the corresponding class is a set. In addition to this empty set, we have, for example, the pair-set. If y and z are sets, then the class composed of y and z contains only sets. As the defining property P(x)=[x=yvx=z] fulfils the condition (c), this class is a set.

With the help of (a) to (d) all the necessary sets can be constructed, except those framed through the Axiom of choice. This axiom may not be considered as strictly belonging to set theory. The requirements from (a) to (d) appear sufficient to avoid the formation of paradoxes in set theory.

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