Certain New Oscillation criteria for Fourth Order Non – Linear Difference Equations

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Abstract

In this paper deals with the oscillatory behaviour of fourth order non – linear difference equation of the form

$$-\Delta^3(p_{n-1}\Delta u_{n-1}) + f_n u_n = 0\ , n\epsilon N(1) \qquad (1.1)$$

Where $f_n = q_n - 3p_n - 3p_{n-1}$. Δ *is the forward difference operator defined by* $\Delta u_n = u_{n+1} - u_n$.*By a solution of (1.1) is consider as* $\{u_n\}$ *exist an N(a) for some* $a \in N$. *Examples are given to illustrate the importance of the results.*

1. Introduction

Consider the non – linear difference equation

$$-\Delta^3(p_{n-1}\Delta u_{n-1}) + f_n u_n = 0\ , n\epsilon N(1) \qquad (1.1)$$

Where $f_n = q_n - 3p_n - 3p_{n-1}$. Δ is the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$.By a solution of (1.1) is consider as $\{u_n\}$ exist an N(a) for some $a \in N$. The function f satisfies the following conditions.

(H1): $\{p_n\}, \{q_n\}$ are real positive sequences, $q_n \neq 0$ for infinitely many values of n.

(H2): $f : R \to R$ is continuous and $xf(x) > 0$ for all $x \neq 0, f(x) \geq 0$

(H3): $\sum_{l=0}^{\infty} \frac{1}{p_l} < \infty$

(H4): $\sum_{l=0}^{\infty} \frac{1}{p_l} = \infty$

Difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, quanta in radiation, genetics in biology, economics, psychology, sociology, etc., unfortunately, these are only considered as the discrete analogs of differential equations. It is an indisputable fact that difference equations appeared much earlier than differential equations and were instrumental in paving the way for the development of the latter. It is only recently that difference equations have started receiving the attention they deserve. Perhaps this is largely due to the advent of computers, where differential equations are solved by using their approximate difference equation formulations. The theory of difference equations has grown at an accelerated pace in the past decade. It now occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

In this study we compared to second and higher order difference equations, the study of third and fourth order difference equations has received less attention. Some fourth order difference equations can be found in [1] to [10]. However, it seems there is very very less known regarding the oscillation of equation (1.1).It is a extended version of [11]. Our aim of in this paper is to present some oscillation criteria for equation (1.1).

2. Main results:

Theorem 2.1 :

If $\{u_n\}$ is a oscillatory solution of equation (1.1) for $n \geq n_1, then\ there\ exists\ a\ n_2 \geq n_1$ such that $\Delta^2(p_{n-1}\Delta u_{n-1}) > 0$.

Proof :

Let $\{u_n\}$ be a non – oscillatory solution of equation (1.1) without loss of generality we may assume that $u_n > 0$,for $n \geq n_1, n_2 \geq n_1$.

From (1.1) we have $\Delta^3(p_{n-1}\Delta u_{n-1}) \leq 0\ for\ all\ n\epsilon N(1)$.We have to prove that $\Delta^2(p_{n-1}\Delta u_{n-1}) > 0$.

Assume that contrary$\Delta^2(p_{n-1}\Delta u_{n-1}) \leq 0$ for $n \geq n_2 and\ n_2 \geq n_1 + 1$. Since $\Delta^2(p_{n-1}\Delta u_{n-1})$ is decreasing such that $\Delta^2(p_{n-1}\Delta u_{n-1}) \leq C\ for\ n \geq n_3$ .Summing the last inequality from $n_3 + 1$to n we get

$$\Delta(p_{n-1}\Delta u_{n-1}) \leq \Delta(p_{n_3+1}\Delta u_{n_3+1}) + C(n - n_3 - 1)$$

Letting $n \to \infty\ then\ \Delta(p_{n-1}\Delta u_{n-1}) \to -\infty$ then there is an integer $n_4 \geq n_3 + 1$ such that $n \geq n_4$, $\Delta(p_{n-1}\Delta u_{n-1}) \leq \Delta(p_{n_4}\Delta u_{n_4}) < 0$

Summing the last inequality from $n_4 + 1\ to\ n$ we get $p_{n-1}\Delta u_{n-1} \leq p_{n_4+1}\Delta u_{n_4+1} + C(n - n_4 - 1)$.

Letting $n \to \infty$ then $p_{n-1}\Delta u_{n-1} \to -\infty$ thus there is an integer $n_5 \geq n_4 + 1$ such that for $n \geq n_5$ $p_{n-1}\Delta u_{n-1} \leq p_{n_5}\Delta u_{n_5} < -C$ which implies that $\Delta u_{n-1} \leq \frac{-C}{p_{n-1}}$. Summing this inequality from $n_5 + 1\ to\ n$

we get $u_{n-1} \leq u_{n_5+1} - C\sum_{t=n_5+1}^{n}\frac{1}{p_{n-1}}$.

Let $n \to \infty$ then $u_{n-1} \to -\infty$, this is contradiction. Therefore $\Delta^2(p_{n-1}\Delta u_{n-1}) > 0$.

Theorem 2.2 :

Assume that the difference equation (1.1) holds the condition (H1) to (H4) and there exist a positive sequence $\{u_n\}_{n=n_0}^{\infty}$ such that

$$\limsup_{n\to\infty}\sum_{\gamma=n_2+1}^{n}\left(\frac{b_\gamma}{p_\gamma\Delta u_\gamma}f_\gamma u_\gamma + \frac{\Delta b_\gamma}{4b_\gamma(\gamma - n_1)}\right) = \infty \qquad (2.1)$$

Then every solution $\{u_n\}$ of equation (1.1) is oscillatory.

Proof :

Let $\{u_n\}$ be a non – oscillatory solution of equation (1.1) without loss of generality, we may assume that $u_n > 0$.From lemma (2.1) it follows that $y_n > 0$, $p_{n-1}\Delta u_{n-1} > 0$, $\Delta(p_{n-1}\Delta u_{n-1})$ *and* $\Delta^2(p_{n-1}\Delta u_{n-1}) > 0$ for $n \geq n_1, n_2 \geq n_1$.we define,

$$v_n = \left(\frac{b_n}{p_n\Delta u_n}\right)\Delta^2(p_{n-1}\Delta u_{n-1}) \qquad (2.2)$$

$$\Delta v_n = \Delta^2(p_n\Delta u_n)\Delta\left(\frac{b_n}{p_n u_n}\right) + \frac{b_n}{p_n\Delta u_n}\Delta^3(p_{n-1}\Delta u_{n-1})$$

$$= \Delta^2(p_n\Delta u_n)\left[\frac{p_n\Delta u_n\Delta b_n - b_n\Delta(p_n\Delta u_n)}{p_n\Delta u_n p_{n+1}\Delta u_{n+1}}\right] + \frac{b_n}{p_n\Delta u_n}\Delta^3(p_{n-1}\Delta u_{n-1})$$

$$= \frac{\Delta b_n}{b_{n+1}}\left(\frac{b_{n+1}\Delta^2(p_n\Delta u_n)}{p_{n+1}\Delta u_{n+1}}\right) - \frac{\Delta^2(p_n\Delta u_n)b_n\Delta(p_n\Delta u_n)}{p_n\Delta u_n p_{n+1}\Delta u_{n+1}} + \frac{b_n}{p_n\Delta u_n}\Delta^3(p_{n-1}\Delta u_{n-1}) \qquad (2.3)$$

Since from the inequality we have

$$p_{n+1}\Delta u_{n+1} \geq p_n\Delta u_n$$

From equation (2.3)

$$\Delta v_n \leq \frac{\Delta b_n}{b_{n+1}}v_{n+1} - \frac{b_n\Delta^2(p_n\Delta u_n)\Delta(p_n\Delta u_n)}{p_n\Delta u_n p_{n+1}\Delta u_{n+1}} + \frac{b_n}{p_n\Delta u_n}\Delta^3(p_{n-1}\Delta u_{n-1}) \qquad (2.4)$$

Consider,

$$\sum_{t=n_1+1}^{n}\Delta^2(p_{t-1}\Delta u_{t-1}) = \Delta(p_{n-1}\Delta u_{n-1}) - (a_{n_1+1}\Delta u_{n_1+1}).$$

$$\Delta(p_{n-1}\Delta u_{n-1}) = a_{n_1+1}\Delta u_{n_1+1} + \sum_{t=n_1+1}^{n}\Delta^2(p_{t-1}\Delta u_{t-1}) \geq$$

$(n - n_1 - 1)\Delta^2(p_{n-1}\Delta u_{n-1})$ For $n \in n + 1$

$$\Longrightarrow \Delta(p_n\Delta u_n) \geq (n - n_1)\Delta^2(p_n\Delta u_n)$$

Therefore

$$\Delta v_n \leq \frac{\Delta b_n}{b_{n+1}}v_{n+1} - \frac{b_n(n-n_1)\,(\Delta^2(p_n\Delta u_n))^2}{p_n\Delta u_n p_{n+1}\Delta u_{n+1}} + \frac{b_n}{p_n\Delta u_n}\Delta^3(p_{n-1}\Delta u_{n-1}) \qquad (2.5)$$

$$p_n\Delta u_n \geq (n - n_1)^2\Delta^2(p_n\Delta u_n)$$

$$\Delta v_n \leq \frac{\Delta b_n}{b_{n+1}} v_{n+1} - \frac{b_n(n-n_1)}{b_{n+1}^2} v_{n+1}^2 + \frac{b_n}{p_n \Delta u_n} f_n u_n$$

Then we have $\Delta v_n \leq \frac{b_n}{p_n \Delta u_n} f_n u_n + \frac{(\Delta b_n)^2}{4b_n(n-n_1)} - \left[\frac{s}{b_{n+1}} - \frac{\Delta b_n}{2s}\right]^2$ where $s = (b_n(n-n_1))^{\frac{1}{2}}$

$$\Delta v_n < \frac{b_n}{p_n \Delta u_n} f_n u_n + \frac{(\Delta b_n)^2}{4b_n(n-n_1)}$$

Summing this inequality from $n_2 + 1$ to n.

We get $v_n \leq v_{n_2} + \sum_{\gamma=n_2+1}^{n} \left(\frac{b_\gamma}{p_\gamma \Delta u_\gamma} f_\gamma u_\gamma + \frac{\Delta b_\gamma}{4b_\gamma(\gamma-n_1)}\right) \sum_{\gamma=n_2+1}^{n} \left(\frac{b_\gamma}{p_\gamma \Delta u_\gamma} f_\gamma u_\gamma + \frac{\Delta b_\gamma}{4b_\gamma(\gamma-n_1)}\right)$ (2.6)

This is contradiction. Hence it is complete the proof.

Corollary : Assume that the difference equation (1.1) holds the condition of theorem 2.2 except the condition 2.1 is replaced by

$$\limsup_{n\to\infty} \sum_{\beta=n_1+1}^{n} \left(\frac{\delta_\beta}{p_\beta \Delta u_\beta}(q_\beta - 3p_\beta - 3p_{\beta-1})u_\beta + \frac{(\beta+1)^\lambda - \beta^\lambda}{4\delta_\beta(\beta - n_0)}\right) = \infty$$

Then every solution of(1.1) is Oscillatory.

Here we apply the double sequence Sn,m in the difference equation (1.1)

Definition:

Let Sn,m be a double sequence of real numbers.

$\lim_{n,m\to\infty} S_{n,m} = \infty$ if for every $\alpha \epsilon R$,there exist $K = K(\alpha)\epsilon N$ such that if n,m $\geq k$ then $S_{n,m} > \alpha$

$\lim_{n,m\to\infty} S_{n,m} = -\infty$ if for every $\beta \epsilon R$,there exist $K = K(\beta)\epsilon N$ such that if n,m $\geq k$ then $S_{n,m} < \beta$

Theorem 2.3 :

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Assume that (H1) to (H4) holds and let {bn} be a positive sequence and assume that there exist a double sequence $\{S_{n,m}; n \geq m \geq 0\}$ such that

Sn,m >0 for n>m>0

$\lim_{n\to\infty} S_{n,m}$=0

$S_{m,m}$= 0 for $m \geq 0$

$$L_2 S_{n,m} = S_{n,m+1} - S_{n,m}$$

Where $L_2 S_{n,m} = S_{n,m+1} - S_{n,m}$

If $\limsup_{n\to\infty} \frac{1}{S_{n,0}} \sum_{m=0}^{n-1} \left[S_{n,m} \frac{b_n}{p_n \Delta u_n} f_n u_n - \frac{b_{n+1}^2}{b_n}\left(S_{n,m} - \frac{\Delta b_n}{b_{n+1}}\sqrt{S_{n,m}}\right)^2\right] = \infty$

(2.7)

Then every solution of equation(1.1) is oscillatory

Proof :

Let $\{u_n\}$ be a non – oscillatory solution of equation (1.1),which may assume to be eventually positive. From equation (2.2) &(2.3) we define the sequence v_n as follows

$$\Delta v_n \leq \frac{\Delta b_n}{b_{n+1}} v_{n+1} - \frac{b_n(n-n_1)}{b_{n+1}^2} v_{n+1}^2 + \frac{b_n}{p_n \Delta u_n} f_n u_n$$

$$\frac{b_n}{p_n \Delta u_n} f_n u_n \geq \Delta v_n - \frac{\Delta b_n}{b_{n+1}} v_{n+1} + \frac{b_n(n-n_1)}{b_{n+1}^2} v_{n+1}^2$$

$$\sum_{m=n_2}^{n-1} S_{n,m} \frac{b_n}{p_n \Delta u_n} f_n u_n \geq \sum_{m=n_2}^{n-1} S_{n,m} \Delta v_n + \sum_{m=n_2}^{n-1} S_{n,m} \frac{b_n(n-n_1)}{b_{n+1}^2} v_{n+1}^2 - \sum_{m=n_2}^{n-1} S_{n,m} \frac{\Delta b_n}{b_{n+1}} v_{n+1} \qquad (2.8)$$

This yields after summing by parts

$$\sum_{m=n_2}^{n-1} S_{n,m} \frac{b_n}{p_n \Delta u_n} f_n u_n \geq S_{n,n_2} v_{n_2} + \sum_{m=n_2}^{n-1} v_{n+1} L_2 S_{n,m} + \sum_{m=n_2}^{n-1} S_{n,m} \frac{b_n(n-n_1)}{b_{n+1}^2} v_{n+1}^2 - \sum_{m=n_2}^{n-1} S_{n,m} \frac{\Delta b_n}{b_{n+1}} v_{n+1}$$

$$= \boldsymbol{S_{n,n_2} v_{n_2}} - \sum_{m=n_2}^{n-1} \frac{\sqrt{\boldsymbol{S_{n,m} b_n (n-n_1)}}}{\boldsymbol{b_{n+1}}} \boldsymbol{v_{n+1}} + \frac{\boldsymbol{b_{n+1}}}{2\sqrt{\boldsymbol{S_{n,m} b_n (n-n_1)}}} \left[\boldsymbol{s_{n,m}} \sqrt{\boldsymbol{S_{n,m}}} - \frac{\Delta \boldsymbol{b_n}}{\boldsymbol{b_{n+1}}} \boldsymbol{S_{n,m}} \right]^2 + \frac{1}{4} \sum_{m=n_2}^{n-1} \frac{\boldsymbol{b_{n+1}^2}}{\boldsymbol{b_n}} \left(\boldsymbol{s_{n,m}} - \frac{\Delta \boldsymbol{b_n}}{\boldsymbol{b_{n+1}}} \sqrt{\boldsymbol{S_{n,m}}} \right)^2$$

$$\frac{1}{4}\sum_{m=n_2}^{n-1} \frac{b_{n+1}^2}{b_n}\left(S_{n,m} - \frac{\Delta b_n}{b_{n+1}}\sqrt{S_{n,m}}\right)^2 \sum_{m=n_2}^{n-1}\left[S_{n,m}\frac{b_n}{p_n \Delta u_n} f_n u_n - \frac{b_{n+1}^2}{b_n}\left(s_{n,m} - \frac{\Delta b_n}{b_{n+1}}\sqrt{S_{n,m}}\right)^2\right] > S_{n,n_2} v_{n_2}$$

$$\frac{1}{4}\sum_{m=n_2}^{n-1} \frac{b_{n+1}^2}{b_n}\left(S_{n,m} - \frac{\Delta b_n}{b_{n+1}}\sqrt{S_{n,m}}\right)^2 \sum_{m=n_2}^{n-1}\left[S_{n,m}\frac{b_n}{p_n \Delta u_n} f_n u_n - \frac{b_{n+1}^2}{b_n}\left(s_{n,m} - \frac{\Delta b_n}{b_{n+1}}\sqrt{S_{n,m}}\right)^2\right] > S_{n,0}\left(v_{n_2} + \sum_{m=0}^{n-1} \frac{b_n}{p_n \Delta u_n} f_n u_n\right)$$

$$\limsup_{n\to\infty} \frac{1}{S_{n,0}} \sum_{m=0}^{n-1} \left[S_{n,m} \frac{b_n}{p_n \Delta u_n} f_n u_n - \frac{b_{n+1}^2}{b_n}\left(s_{n,m} - \frac{\Delta b_n}{b_{n+1}}\sqrt{S_{n,m}}\right)^2 \right] > \infty \qquad (2.9)$$

Which clearly contradicts (2.7).

Remarks :

By choosing the sequence $\{S_{n,m}\}$ in appropriate manners, we can derive several oscillation criteria for(1.1).

Let us consider the double sequence $\{S_{n,m}\}$ defined by

$$S_{n,m} = (n-m)^\lambda, \lambda \geq 1, n \geq m \geq 0$$

$$S_{n,m} = (n-m)^\lambda, \lambda > 2, n \geq m \geq 0$$

$$where\ (n-m)^\lambda = (n-m)(n-m+1)\ldots(n-m+\lambda-1)$$

Corollary: Assume that all the assumptions of Theorem 2.3 hold except that condition (2.7) is replaced by

$$\limsup_{n\to\infty} \frac{1}{log(n+1)\lambda} \sum_{m=0}^{n-1} \left[log\left(\frac{n+1}{m+1}\right)^\lambda \frac{b_n}{p_n \Delta u_n} f_n u_n - \frac{b_{n+1}^2}{b_n}\left(log\left(\frac{n+1}{m+1}\right)^\lambda - \frac{\Delta b_n}{b_{n+1}} log\left(\frac{n+1}{m+1}\right)^{\frac{\lambda}{2}} \right)^2 \right] = \infty$$

Then every solution of equation (1.1) is oscillatory.

Example : 1

Consider the difference equation

$$\Delta^3\big((n-1)\Delta u_{n-1}\big) - 4(3n+1)u_n = 0$$

Satisfies all the condition of Theorem 2.2 &2.3 for Sn,m=n+m & $S_{n,m} = \frac{1}{\sqrt{n+m}}$. Hence all the solution of equation (1.1) are oscillatory.

3. References

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