CONDITIONS ON DRℓ - GROUP

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Abstract: In this paper we introduce that an ℓ - group, Brouwerian Algebra and Boolean ring can be realized from a DRℓ - group by some specialization.

Key words: ℓ - group, Brouwerian Algebra, Boolean ring, Residuated lattice, DRℓ - group.

1. Preliminaries

Definition 1.1 [4]

A non – empty set G is called an ℓ – group if and only if

(i) (G, +) is a group

(ii) (G, $\leq$) is a lattice

(iii) If $x \leq y$, then $a + x + b \leq a + y + b$, for all a, b, x, y in G.

(or)

$(a + x + b) \vee (a + y + b) = (a + x \vee y + b)$

$(a + x + b) \wedge (a + y + b) = (a + x \wedge y + b)$, for all a, b, x, y in G.

Definition 1.2 [4]

An ℓ – group G is called a commutative ℓ – group if a + b = b + a, for all a, b in G.

Definition 1.3 [4], [5]

A lattice L is called a residuated lattice if

(i) (L, **.**) is an ℓ – group.

(ii) Given a, b in L, there exist the largest x, y such that

$bx \leq a$ and $yb \leq a$.

Definition 1.4 [1], [4]

A non – empty set B is called a Brouwerian Algebra if and only if

(i) $(B, \leq)$ is a lattice

(ii) B has a least element

(iii) To each a, b in B, there is a least $x = a - b$ in B such that $b \vee x \geq a$

Definition 1.5 [4]

A ring $(R, +, .)$ is called a Boolean ring if and only if $a . a = a$, for all a in R.

Definition 1.6 [4]

A system $A = (A, +, \leq)$ is called dually residuated lattice ordered group (simply DRℓ – group) if and only if

(i) $(A, +)$ is an abelian group.

(ii) $(A, \leq)$ is a lattice.

(iii) $b \leq c \Rightarrow a + b \leq a + c$, for all a, b, c in A

(iv) Given a, b in A, there exist a least element $x = a - b$ in A such that $b + x \geq a$.

Definition 1.7 [4]

A system $A = (A, +, \vee, \wedge)$ is called a DRℓ – group if and only if

(i) $(A, +)$ is an abelian group.

(ii) $(A, \vee, \wedge,)$ is a lattice.

(iii) $a + (b \vee c) = (a + b) \vee (a + c)$,

$a + (b \wedge c) = (a + b) \wedge (a + c)$, for all a, b, c in A.

(iv) $x + (y - x) \geq y$,

$x - y \leq (x \vee z) - y$,

$(x + y) - y \leq x$, for all x, y, z in A.

Remark [4]

Two definitions for DRℓ – group are equivalent.

Examples 1.1 [4]

Commutative ℓ - group, Brouwerian Algebra and Boolean ring are DRℓ - groups.

Properties of a DRℓ - group

Now we see the properties of a DRℓ - group which is already established in [4] .

1. $[(a - b) \vee 0] + b = a \vee b$,
2. $a \leq b \Rightarrow a - c \leq b - c$ and $c - b \leq c - a$,
3. $(a \vee b) - c = (a - c) \vee (b - c)$,
4. $a - (b \wedge c) = (a - b) \vee (a - c)$,
5. $a - (b \vee c) = (a - b) \wedge (a - c)$,
6. $(b \wedge c) - a = (b - a) \wedge (c - a)$,
7. $a \geq b \Rightarrow (a - b) + b = a$,
8. $a \vee b + a \wedge b = a + b$,
9. $(a - b) \vee 0 + a \wedge b = a$,
10. $a \vee b - a \wedge b = (a - b) \vee (b - a)$,
11. $a - (b - c) \leq (a - b) + c$ and $(a + b) - c \leq (a - c) + b$, for all a, b, c in A.

Theorem 1.2 [4]

The direct product of the DRℓ - group is also a DRℓ - group.

Theorem 1.3[4]

Any DRℓ – group is a distributive lattice.

2. Realization from a DRℓ - group

Theorem : 2.1

If A is a DRℓ – group and $a + b = a \vee b$, for all a, b in A, then A is a Brouwerian Algebra.

Proof :

Given A is a DRℓ – group and a + b = a $\vee$ b, for all a, b in A.

To prove A is a Brouwerian Algebra.

By given, we have

(i) (A, $\leq$) is a lattice

(ii) A has a least element 0.

(iii) To each a, b in A, there exist a least element x in A such that $b \vee x = b + x \geq a$.

Hence A is a Brouwerian Algebra.

Theorem : 2.2

If A is a DRℓ – group and (A, $\leq$, –) is a Brouwerian Albegra, then $a + b = a \vee b$, for all a, b in A.

Proof :

Given A is a DRℓ – group and a Brouwerian Algebra.

To prove $a + b = a \vee b$, for all a, b in A.

Let a, b in A be arbitrary.

$\Rightarrow$ there exist a least element x = a – b in A such that $b \vee x \geq a$.

We have $y \vee (x-y) = y \vee x$ in any Brouwerian Algebra $\rightarrow$ (1)

$\Rightarrow a \vee (a - a) = a \vee a$

$\Rightarrow a \vee a \quad = a$

$\Rightarrow$ 0 is the least element.

By property 8, we have

$a + b \quad = a \vee b + a \wedge b$

$\geq a \vee b + 0 \quad = a \vee b$

$\Rightarrow a + b \geq a \vee b \rightarrow (2)$

By property 8 $\Rightarrow$ $a + b - a \vee b = a \wedge b \leq a \leq a \vee b$

$$\Rightarrow \quad a \vee b = (a \vee b) \vee [(a + b) - (a \vee b)]$$

$$= (a \vee b) \vee (a + b), \text{ by } (1)$$

$$= a + b, \text{ by } (2)$$

Proposition : 2.1

If A is a DRℓ – group, then

(i) $a * b \geq 0$

(ii) $a * b = 0 \Leftrightarrow a = b$

(iii) $a * b = b * a$

(iv) $(a \vee b) * (a \wedge b) = a * b$, for all a, b in A.

Proof :

Let a, b in A be arbibrary.

(i) By property 3.10, we have

$$a \vee b - a \wedge b = (a - b) \vee (b - a)$$

$$= a * b$$

$\Rightarrow a * b = a \vee b - a \wedge b \qquad \rightarrow (1)$

we have $a \vee b \geq a \wedge b$, for all a, b in A.

$\Rightarrow a \vee b - a \wedge b \geq 0 \qquad \rightarrow (2)$

Using (2) in (1), we get

$a * b \geq 0$.

(ii) Assume that $a * b = 0$

To prove $a = b$

Now $a * b = 0 \Rightarrow a \vee b - a \wedge b = 0$, by (1)

$$\Rightarrow a \vee b = a \wedge b$$

$\Rightarrow$ a = b.

Conversely, assume that a = b

To prove that a $\ast$ b = 0

Now, a = b $\Rightarrow$ a $\vee$ b = a $\wedge$ b

$\Rightarrow$ a $\vee$ b – a $\wedge$ b = 0

$\Rightarrow$ a $\ast$ b = 0, by (1)

(iii) a $\ast$ b = (a - b) $\vee$ (b - a)

= (b - a) $\vee$ (a - b)

= b $\ast$ a, for all a, b in A.

(iv) (a $\vee$ b) $\ast$ (a $\wedge$ b) = [(a $\vee$ b) – (a $\wedge$ b)] $\vee$ [(a $\wedge$ b) – (a $\vee$ b)]

= [(a - b) $\vee$ (b - a)] $\vee$ [(b - a) $\vee$ (a - b)], by property 10 [4]

= (a - b) $\vee$ (b - a)

= a $\ast$ b, for all a, b in A.

Theorem 2.3

If the symmetric difference is associative in a DRℓ – group A, then (A, *, $\wedge$) is a Boolean ring and further

a + b = a $\vee$ b = a * b * (a $\wedge$ b)

a - b = a * (a $\wedge$ b), for all a, b in A.

Proof :

Given that the symmetric difference is associative in a DRℓ – group A.

To prove (1) (A, *, $\wedge$) is a Boolean ring.

(2) a + b = a $\vee$ b = a * b * (a $\wedge$ b),

a – b = a * (a $\wedge$ b), for all a, b in A.

For (1) :

(i) For all a, b in A ⇒ a * b in A:

Let a, b in A be arbitrary.

Then $a * b = (a - b) \vee (b - a)$

Now a, b in A ⇒ a – b, b – a in A

$\Rightarrow (a - b) \vee (b - a)$ in A

⇒ a * b in A.

(ii) a * (b * c) = (a * b) * c, for all a, b, c in A:

This follows by assumption.

(iii) There exist an element 0 in A such that a * 0 = 0 * a = a, for all a in A:

Let a in A be arbitrary.

Since A is a DRℓ – group we have 0 in A.

By associativity of *, we have

$$a * (a * 0) = (a * a) * 0$$

$$= 0 * 0$$

$$= 0$$

Therefore a * 0 = a, by proposition 2.1

$$(a * 0) * a = (0 * a) * a$$

$$= 0 * (a * a)$$

$$= 0 * 0 = 0$$

Therefore 0 * a = a, by proposition 2.1

(iv) To each a in A, there exist an element a in A such that a * a = 0:

Let a in A be arbitrary.

Then $a * a = (a - a) \vee (a - a) = 0$

Thus $a * a = 0$

(v) $a * b = b * a$, for all a, b in A:

Let a, b in A be arbitrary.

Then $a * b = b * a$, for all a, b in A, by proposition 2.1

Therefore (A, *) is an abelian group.

(vi) For all a, b in A $\Rightarrow a \wedge b$ in A:

Let a, b in A be arbitrary.

Then $a \wedge b$ in A, since A is a DRℓ – group

Thus a, b in A $\Rightarrow a \wedge b$ in A.

(vii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all a, b, c in A :

Let a, b, c in A be arbitrary.

Then $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all a, b, c in A, since A is a DRℓ – group

(viii) $a \wedge (b * c) = (a \wedge b) * (a \wedge c)$:

$(b * c) \wedge a = (b \wedge a) * (c \wedge a)$, for all a, b, c in A :

Let a, b, c in A be arbitrary.

For any x, y we have

$(x \vee y) * (x \wedge y) = x * y$, by proposition 2.1 (1)

put $x = b – a$ and $y = a$ in (1), we have

$$[a \vee (b – a)] * [a \wedge (b – a)] = a * (b – a)$$

Pre-multiply $a \vee (b – a)$ on both sides, we get

$$[a \vee (b - a)] * ([a \vee (b - a)] * [a \wedge (b - a)]) = [a \vee (b - a)] * [a*(b - a)]$$

$\Rightarrow ([a\vee (b - a)] * [a \vee (b - a)]) * [a \wedge (b - a)] = [a \vee (b - a)] * [a*(b - a)]$,

by associative law

$\Rightarrow 0 * [a \wedge (b - a)] = [a \vee (b - a)] * [a *(b - a)]$, since $a * a = 0$

$\Rightarrow [a \wedge (b - a)] = [a \vee (b - a)] * [a *(b - a)]$, since $a * 0 = a \quad \rightarrow (2)$

But $[a * (b - a)] \quad = [a \vee (b - a)] - [a \wedge (b - a)]$

$= (a \vee b) - [a \wedge (b - a)]$

$= [(a \vee b) - a] \vee [(a \vee b) - (b - a)]$, by property 4

$= (b - a) \vee [(a \vee b) - (b - a)]$ by property 3

$= (a \vee b) - (b - a)$

$= a \vee b$

(ie) $a * (b - a) = a \vee b \rightarrow (3)$

Using (3) in (2), we get

$a \wedge (b - a) \quad = [a \vee (b - a)] * (a \vee b)$

$= (a \vee b) * (a \vee b) = 0$, since $a * a = 0$

Hence $a \wedge (b - a) = 0 \rightarrow (4)$

so that $[a \wedge (b - c)] - [(a \wedge b) - c]$

$= (a - [(a \wedge b) - c]) \wedge ((b - c) - [(a \wedge b) - c])$ by property 6

$= (a - [(a - c) \wedge (b - c)]) \wedge [(b - c) - (a - c)]$ by property 6 and $a \leq b \Rightarrow a \wedge b = a$

$= (a - [(a - c) \wedge (b - c)]) \wedge [b - (c \vee a)]$

$= (a-[(a - c) \wedge (b - c)]) \wedge [(b - c) \wedge (b - a)]$, by property 5

$< a \wedge (b - a) \wedge (b - c)$

$= 0 \wedge (b - c)$, by (4)

$= 0$

so that $a \wedge (b - c) < (a \wedge b) - c$

But we always have $(a \wedge b) - c < a \wedge (b - c)$,

since $(a \wedge b) - c = (a - c) \wedge (b - c) < a \wedge (b - c)$

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Hence $a \wedge (b - c) = (a \wedge b) - c \quad \rightarrow(5)$

Now, $a \wedge (b * c) = a \wedge [(b - c) \vee (c - b)]$

$= [a \wedge (b - c)] \vee [a \wedge (c - b)]$

$= [(a \wedge b) - c] \vee [(a \wedge c) - b]$, by (5)

$= ([(a \wedge b) - a] \vee [(a \wedge b) - c]) \vee ([(a \wedge c) - a] \vee [(a \wedge c) - b])$,

since $0 = a \wedge (b - a) = (a \wedge b) - a$

$0 = a \wedge (c - a) = (a \wedge c) - a$

$= [(a \wedge b) - (a \wedge c)] \vee [(a \wedge c) - (a \wedge b)]$

$= (a \wedge b) * (a \wedge c)$

Thus $a \wedge (b*c) = (a \wedge b) * (a \wedge c)$, for all a, b, c in A.

Also, $(b * c) \wedge a = [(b - c) \vee (c - b)] \wedge a$

$= a \wedge [(b - c) \vee (c - b)]$

$= [a \wedge (b - c)] \vee [a \wedge (c - b)]$

$= [(a \wedge b) - c] \vee [(a \wedge c) - b]$

$= ([(a \wedge b) - a] \vee [(a \wedge b) - c]) \vee ([(a \wedge c) - a] \vee [(a \wedge c) - b])$,

since $0 = a \wedge (b - a) = (a \wedge b) - a$

$0 = a \wedge (c - a) = (a \wedge c) - a$

$= [(a \wedge b) - (a \wedge c)] \vee [(a \wedge c) - (a \wedge b)]$

$= [(b \wedge a) - (c \wedge a)] \vee [(c \wedge a) - (b \wedge a)]$

$= (b \wedge a) * (c \wedge a)$

Thus $(b * c) \wedge a = (b \wedge a) * (c \wedge a)$, for all a, b, c, in A.

(ix) $a \wedge a = a$, for all a in A:

Let a in A be arbitrary.

Then $a \wedge a = a$, for all a in A, since A is a $DR\ell$ – group.

Thus $(A, *, \wedge)$ is a Boolean Algebra.

For (2):

Let a, b in A be arbitrary.

Then $a * (a \wedge b) = [a \vee (a \wedge b)] - [a \wedge (a \wedge b)]$

$= a - (a \wedge b)$, by absorption and associative laws

$= (a - a) \vee (a - b)$, by property 5

$= 0 \vee (a - b) = a - b$

Thus $a - b = a * (a \wedge b)$, for all a, b in A.

Now, $a * 0 = a \Rightarrow a \geq 0$.

so that $a + a \geq a$, for every a and $0 - a \leq 0$, for every a.

Also, $a + a = (a + a) * 0$

$= (a + a) * (a * a)$

$= [(a + a) * a] * a$

$= ([(a + a) \vee a] - [(a + a) \wedge a]) * a$, by property 10

$= [(a + a) - a] * a$, since $a * a \geq a$.

$= ([(a + a) - a] - a) \vee (a - [(a + a) - a])$

$= a - [(a + a) - a]$

Hence $(a + a) - a = (a - [(a + a) - a]) - a$

$= (a - a) - [(a + a) - a]$

$= 0 - [(a + a) - a]$

$= 0 - a \leq 0$

$\Rightarrow a + a \leq a$

Hence $a + a = a$

Now, $(a + b) - (a \vee b) = (a + b) - [(a \vee b) + (a \vee b)]$

$\leq (a - [(a \vee b) + (a \vee b)]) + b$, by property 11

$= [a - (a \vee b)] - (a \vee b) + b$

$= [a - (a \vee b)] + [b - (a \vee b)]$

$\leq 0 + 0 = 0$

so that $a + b \leq a \vee b \rightarrow (6)$

Since $a \geq 0$ and $b \geq 0$ and by property 8, we have

$a + b = a \vee b + a \wedge b$

(ie) $a + b \geq a \vee b \rightarrow (7)$

From (6) and (7), we have

$a + b = a \vee b$

Now, $a * b * (a \wedge b) = a * [b * (b \wedge a)]$

$= a * (b - a)$, by property 10 and by property 5

$= [a \vee (b - a)] - [a \wedge (b - a)]$

$= (a \vee b) - 0$, by (4)

$= a \vee b = a + b$

Thus $a + b = a \vee b = a * b * (a \wedge b)$, for all a, b in A.

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