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Cubic Spline Interpolation with New Conditions on M_0 and M_n

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Abstract

In this communication, we defined new conditions on M_0 and M_n of cubic spline interpolation. With the defined conditions, an attempt is made to investigate the accuracy of the estimation of the dependent variable at particular values of the independent variable. With these new conditions we observed that the error is reduced considerably compared to the other types of conditions on M_0 and M_n.

1. Introduction

Any function which would effectively correlate the data would be difficult to obtain and highly unwieldy. To this end, the idea of the cubic spline was developed. Using this process, a series of unique cubic polynomials are fitted between each of the data points, with the stipulation that the curve obtained were continuous and appear smooth. These cubic splines can then be used to determine rates of change and cumulative change over an interval.

The first mathematical reference to splines was made in the year 1946 in an interesting paper by Schoenberg (Schoenberg [1]),which is probably the first place that the word "spline" is used in connection with smooth, piecewise polynomial approximation. Generally I.J. Schoenberg is regarded as the father ofsplines, particularly on account of his pioneering paper [2]. However, the ideas have their roots in the aircraft and shipbuilding industries. Splines are types of curves, originally developed for shipbuilding in the days before computer modelling. Naval architects needed a way to draw a smooth curve through a set of points. The solution was to place metal weights (called knots) at the control points, and bend a thin metal or wooden beam (called a spline) through the weights. Through the advent of computers, splines have gained more importance. They were first used as a replacement for polynomials in interpolation and then as a tool to construct smooth and flexible shapes in computer graphics.

In late 1960's, there were no more than a handful of articles mentioning spline functions by name. Some of the papers which have made great contributions in the development of splines include (Loscalzo and Talbot [3], Maclaren [4], Rubin and Khosla[5],Sastry[6], Schoenberg [2]). Convergence properties of the cubic spline method are discussed by Ahlberg and Nilson [7]. Univariate splines were studied intensely in the 60s, and by the mid-70s they were sufficiently well understood to permit a fairly comprehensive treatment in books from. Some of the books which discuss splines thoroughly include (Ahlberg et al. [8], deBoor [9], Prenter [10], Schumaker [11], Shikin and Plis [12], Spath [13]). Some of the earliest papers using spline functions for the smooth approximate solution of ordinary and partial differential equations (PDEs) include (Albasiny and Hoskins [14], Bickely [15], Crank and Gupta [16], Jain and Aziz [17], Jain and Aziz [18], Rubin and Khosla [19], Usmani [20], Usmani and Sakai [21], Usmani and Warsi [22], Rama Chandra Rao [23], Kalyani and Rama Chandra Rao [24]) . These papers demonstrate the approximate methods of solving second, third, fourth, fifth order linear boundary-value problems (BVPs) and solution of elliptic and parabolic equations by spline functions of various degrees.

Today, there are number of research articles published on this subject, and yet it remains an active research area. In these papers various techniques are used such as quadratic, cubic, quartic, quintic, sextic, septic and higher degree splines, and have been discussed for the numerical solution of linear and nonlinear BVPs. A survey of recent spline techniques for solving boundary value problems in ordinary differential equations using cubic, quintic and sextic polynomial andnon-polynomial splines are given in Kumar and Srivastava [25].Splines have many applications in the numerical solution of a variety of problems in mathematics and engineering. Some of them are, fitting of curves, function approximation, solution of integro-differential equations, optimal control problems, computer-aided geometric design, and wavelets and so on. Also, these are useful to solve different problems in atomic and molecular physics, and are used extensively at Boeing and throughout much of the industrial world. The main task of cubic spline interpolation techniques is to find the spline function.We will discuss splines which interpolate equally spaced data points, by taking M_0 as the slope of the line joining the initial point and next immediate point to it and M_n is taken as the slope of the line joining last point and its immediate preceding point, and compared with natural cubic splines by taking different step lengths. Computed errors with End-point-slope and Type I conditions. It is observed that the error is reduced with end-point-slope conditions. Further by taking h is smaller erroris minimized, probably by taking the slopes at the initial and terminalpoints since the spline is known to be the rate of change of tangent (curvature).

2. Cubic Splines

Suppose$(x_i, y_i)$for$i = 0,1,2, \ldots \ldots, n$ be the set of points of known or unknown $y = f(x)$,

where$a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $h_i = x_i - x_{i-1}$, $i = 1,2, \ldots, n$.(1)

The cubic spline

$s_i(x)$ defined in the interval $[x_{i-1}, x_i]$ will satisfy the properties

$s_i(x)$ is almost a cubic in each subinterval $[x_{i-1}, x_i]$, $i = 1,2, \ldots, n$,

$s_i(x_i) = y_i$, $i = 0,1,2, \ldots, n$,

$s_i(x)$, $s'_i(x)$ and $s''_i(x)$ are continuous in $[x_0, x_n]$,

The spline function can be obtain from the following equation given by [6]

$$s_i(x) = \frac{1}{h_i}\left[\frac{(x_i - x)^3}{6} M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i +\right.$$

yi−1−hi26Mi−1xi−x+yi−hi26Mix−xi−1(2)

where $s'_i(x_i) = m_i$, and $s''_i(x_i) = M_i$

In this equation, the spline second derivativesM''_i, are still not known. We use the condition of continuity of $s'_i(x)$ to obtain the recurrence relation

$$\frac{h_i}{6} M_{i-1} + \frac{1}{3}(h_i + h_{i+1}) M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} (i = 1,2, \ldots, n-1) (3)$$

For equal intervals we have $h_i = h_{i+1} = h$ and eq. (4) simplifies to

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i+1} - 2y_i +$$

yi−1, i=1,2,….,n−1.(4)

Equations (4) constitute a system of $(n-1)$ equations in $(n+1)$ unknowns $M_0, M_1, \ldots . . M_n$.To obtain a solution for $M_i's$, we have to impose two new conditions.

2.1 Conditions on M_0 and M_n

The following conditions are defined on M_0 and M_n

$$M_0 = \frac{Y_1 - Y_0}{X_1 - X_0} (5)$$

$$M_n = \frac{y_n - y_{n-1}}{X_n - X_{n-1}} (6)$$

We call these conditions as “End-point-slope” conditions.

The following types of conditions are specified in [24] forM_0 and M_n.

2.2 Type I (Natural cubic splines)

This spline type includes the stipulation that the second derivative be equal to zeroat the endpoints.That is $M_0 = M_n = 0$(6.1)

This results in the spline extending as a line outside the endpoints.

2.3 Type Π(Parabolic Runout Spline)

The parabolic spline imposes the condition that the second derivative at the endpoints, M_0 and M_n be equal to M_1 and M_{n-1}respectively.

That is $M_0 = M_1$, $M_n = M_{n-1}$(6.2)

2.4 Type III (Cubic Runout Spline)

This type of spline has the most extreme endpoint behaviour. It assigns M_0 to be $2M_1 - M_2$and M_n to be $2M_{n-1} - M_{n-2}$i.e.

$M_0 = 2M_1 - M_2$, $M_n = 2M_{n-1} - M_{n-2}$(6.3)

There are many other types of interpolating spline curves, such as the periodic spline and the clamped Spline. The one compared with this work which we have chosen to examine; is not intrinsically superior to, or more widely used than these other types of splines.

When $M_i's$ are known, eq. (2) gives the required cubic spline in the subinterval$[x_{i-1}, x_i]$.

3 Numerical results

We consider certain problems with known functions which facilitate to study the accuracy of estimation using the end-point-slope conditions given by (5) and (6). Further, the resultsobtained through (5) and (6) are compared with the results of Type I. The results are shown in the tabular form. The approximate values by end-point-slope conditions, by Type I conditions and exact values are shown graphically.

Example1.

We consider a function y defined on [1, 6] and suppose that the data of x and y is as follows

$x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3, x_5 = 3.5, x_6 = 4, x_7 = 4.5, x_8 = 5, x_9 = 5.5, x_{10} = 6$ (7)

and $y_0 = 3, y_1 = 9.09375, y_2 = 33,$
$y_3 = 98.15625, y_4 = 243, y_5 = 524.71$
$y_6 = 1023, y_7 = 1843.781, y_8 = 3123,$
$y_9 = 5030.344, y_{10} = 7773$(8)

From (4) we have a system of equations in $M_i's$ (for $i = 1$ to 9)

$$M_0 + 4M_1 + M_2 = 427.5 \quad M_1 + 4M_2 + M_3 = 990$$
$$M_2 + 4M_3 + M_4 = 1912.5$$
$$M_3 + 4M_4 + M_5 = 3285$$
$$M_4 + 4M_5 + M_6 = 5197.5 \quad M_5 + 4M_6 + M_7 = 7740$$
$$M_6 + 4M_7 + M_8 = 11002.5$$
$$M_7 + 4M_8 + M_9 = 15075$$
$$M_8 + 4M_9 + M_{10} = 20047.5$$

Cubic spline with end-point-slope conditions

From (5) and (6) we have

$M_0 = 12.1875, M_{10} = 5485.313$(9)

Substituting (9) in the above system of equations, andsolving we get$M_1 = 65.16567, M_2 = 154.6498, M_3 = 306.2352, M_4 = 532.9094, M_5 = 847.1271,\ M_6 = 1276.082, M_7 = 1788.544, M_8 = 2572.242, M_9 = 2997.487$ (10)

From (2) and from the equations (7) - (10) we get the interpolating polynomial in $1 \le x \le 1.5$ which is

$s_1(x) = 4.0625(1.5 - x)^3 + 21.72189(x - 1)^3 + 4.984375(1.5 - x) + 12.75703(x - 1)$ (11)

Proceeding as the method described above, interpolated functions with end-point-slope conditions are obtained in the subintervals$[x_{i-1}, x_i]$, for $i = 1,2 \ldots,10$ and they are shown in the Table.1.

Cubic spline with Type I

From (6.1) we have $M_0 = 0, M_{10} = 0.$ (12)

Substituting (12) in the system of equations in $M_i^{'}s$ (for $i = 1$ to 9), and solving the obtained system we get

$M_1 = 68.46756, M_2 = 153.6297, M_3 = 307.0135,\ M_4 = 530.8165. M_5 = 854.7203, M_6 = 1247.802, M_7 = 1894.071, M_8 = 2178.414,\ M_9 = 4467.271$(13)

From (2) and from the equations (7), (8), (12) and(13)we get the interpolating polynomial in $1 \le x \le 1.5$ which is

$S_1(x) = 22.82252(x - 1)^3 + 6(1.5 - x) + 12.48187(x - 1)$(14)

The interpolated functions of Type I. in the intervals $[x_{i-1}, x_i]$, for $i = 1,2 \ldots,10$. are given inthe Table.2.Thedata considered follows the function

$y(x) = x^5 - x + 3$(15)We consider the values of $y(x)$ in intervals of 0.05 from $x = 1$ to 1.5 and then interpolate for x using the cubic spline with end-point-slope conditions (5), (6), and by natural cubic splines (6.1).The cubic spline values obtained by thesetwo types of conditions in the interval [1, 1.5] are shown in the Table.3 with their corresponding errors and exact values (15). A comparison is givenin Fig.1and comparison of errors for Ex.1 isshown in Fig.2. The cubic spline values obtained by these two types of conditions in the intervals $[x_{i-1}, x_i]$, for $i = 1,2 \ldots,10$ are shown in the Table.4 with their corresponding errors and exact values.

Example 2

Suppose the data of (x_i, y_i) for $i = 0,1,2,3,4,5,6,7,8,9$ $(n = 10)$ is as given below with interval of differencing h = 0.5

$x_0 = 1, x_1 = 1.5, x_2 = 2,\ x_3 = 2.5, x_4 = 3, x_5 = 3.5, x_6 = 4, x_7 = 4.5, x_8 = 5, x_9 = 5.5,\ x_{10} = 6$ (16)

$y_0 = 1, y_1 = 2.426091, y_2 = 4.30103,\ y_3 = 6.64794, y_4 = 9.477121, y_5 = 2.79407, y_6 = 16.60206, y_7 = 20.90321,\ y_8 = 25.69897, y_9 = 30.99036, y_{10} = 36.77815$ (17)

From (4) we have a system of equations in $M_i^{'}s$ (for $i = 1$ to 9)

$M_0 + 4M_1 + M_2 = 10.77234 \quad M_1 + 4M_2 + M_3 = 11.32731$

$M_2 + 4M_3 + M_4 = 11.57451$

$M_3 + 4M_4 + M_5 = 11.70637$

$M_4 + 4M_5 + M_6 = 11.78508 \quad M_5 + 4M_6 + M_7 = 11.83585$

$M_6 + 4M_7 + M_8 = 11.87052$

$M_7 + 4M_8 + M_9 = 11.89524$

$M_8 + 4M_9 + M_{10} = 11.9135$

Cubic spline with end-point-slope conditions

From (5) and (6) we have

$M_0 = 2.852183, M_{10} = 11.57558.$ (18)

Substituting (18) in the above system of equations and solving the obtained system we get$M_1 = 1.484023, M_2 = 1.984066, M_3 = 1.907023, M_4 = 1.96235, M_5 = 1.949946, M_6 = 2.022947, M_7 = 1.794117, M_8 = 2.671104, M_9 = -0.5833$(19)From (2) and from the equations (16) - (19) we get the interpolating polynomial in $1 \le x \le 1.5$ which is

$s_1(x) = 0.950728\ (1.5 - x)^3 + 0.494674(x - 13+1.7623181.5-x+4.728514x-1$The interpolated functions with end-point-slope conditions in the intervals $[x_{i-1}, x_i]$, for$i = 1,2 \ldots,10.$ are shown in the Table 5.

Cubic spline with Type I

From (6.1) we have $M_0 = 0, M_{10} = 0$ (20)

Taking (20) in the above system of equations and solving the obtained system we get

$M_1 = 2.24834, M_2 = 1.778982, M_3 = 1.963041, M_4 = 1.943364. M_5 = 1.969874, M_6 = 1.962222, M_7 = 2.01709, M_8 = 1.83994, M_9 = 2.51839$(21)

Proceeding as in example1 we get the interpolating polynomial in [1,1.5] which is

$s_1(x) = 0.749447(x - 1)^3 + 2(1.5 - x) + 4.664821(x - 1)$ (22)

The tabulated function for the given data is

$y(x) = x^2 + \log x$ (23)

The interpolated functions are shown in the Table.6 in the intervals $[x_{i-1}, x_i]$, for $i = 1,2 \ldots,10$.we consider the values of $y(x)$ in intervals of 0.05 from $x = 1$ to 1.5 and then interpolate for x using the cubic spline with end-point-slope condition (5) and (6), and by natural cubic splines(6.1). The cubic spline values obtained by these two types of conditions in the interval [1, 1.5] are shown in Table.7 with their corresponding errors and exact values (23). A comparison is given in Fig.3 and error graphs by two types of conditions at n=10 for Ex.2 are shown in Fig.4 respectively. The cubic spline values obtained by these two types of conditions in the intervals $[x_{i-1}, x_i]$, for $i = 1,2 \dots,10$ are shown in Table.8 with their corresponding errors and exact values. The absolute errors with end-point-slope conditions for examples 1 and 2 at h=0.5 are compared with Type I conditions in Table 9.

4 Conclusions

In the present work we applied the cubic spline interpolation method with different types of conditions on M_0 and M_n to approximate the functions at different values of x in examples 1 and 2.The results are given in the tabular for, and shown graphically. From the numericalcomputations it is observed that the error in estimation of y is considerably reduced by the end-point-slope conditions. Further, it is observed that as h is decreasing the estimate is very close to the exactvalue.

Table 1: Spline functions with end-point-slope conditions in the corresponding intervals

Interval	Cubic spline function
[1,1.5]	$4.0625\,(1.5-x)^3+21.72189\,(x-1)^3+4.98437(1.5-x)+12.75703(x-1)$
[1.5,2]	$21.72189\,(2-x)^3+51.54993\,(x-1.5)^3+12.75703(2-x)+53.11252(x-1.5)$
[2,2.5]	$51.54993\,(2.5-x)^3+102.0784\,(x-2)^3+53.11252(2.5-x)+170.7929(x-2)$
[2.5,3]	$102.0784(3-x)^3+177.6365(x-2.5)^3+170.7929(3-x)+441.5909(x-2.5)$
[3,3.5]	$177.6365(3.5-x)^3+282.3757(x-3)^3+441.5909(3.5-x)+978.8436(x-3)$
[3.5,4]	$282.3757(4-x)^3+425.3607(x-3.5)^3+978.8436(4-x)+1939.66(x-3.5)$
[4,4.5]	$425.3607(4.5-x)^3+596.1813(x-4)^3+1939.66(4.5-x)+3538.517(x-4)$
[4.5,5]	$596.1813(5-x)^3+857.414(x-4.5)^3+3538.517(5-x)+6031.647(x-4.5)$
[5,5.5]	$857.414(5.5-x)^3+999.1624(x-5)^3+6031.647(5.5-x)+9810.897(x-5)$
[5.5,6]	$999.1624(6-x)^3+1828.438(x-5.5)^3+9810.897(6-x)+15088.89(x-5.5)$

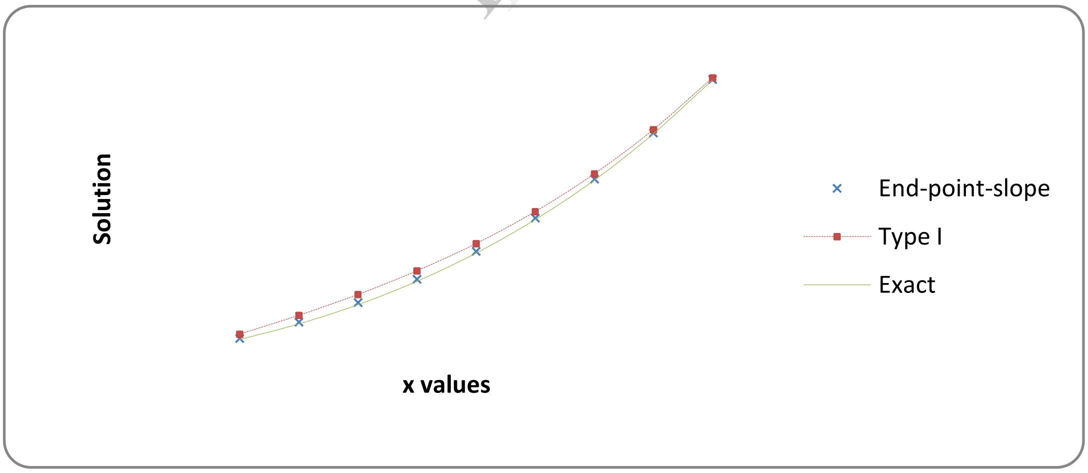

Fig.1: Comparison of approximate values and exact values for Ex.1

Table 2: Spline functions with Type I conditions in the corresponding intervals

Interval	Cubic spline function
[1,1.5]	11.41126 (x-1)3+3(1.5-x) + 6.240935 (x-1)
[1.5,2]	$11.41126\ (2-x)^3 + 25.60496\ (x-1.5)^3 + 6.240935\ (2-x) + 26.59876\ (x-1.5)$
[2,2.5]	$25.60496\ (2.5-x)^3 + 51.16891\ (x-2)^3 + 26.59876\ (2.5-x) + 85.36402\ (x-2)$
[2.5,3]	$51.16891\ (3-x)^3 + 88.46942\ (x-2.5)^3 + 85.36402\ (3-x) + 220.8826\ (x-2.5)$
[3,3.5]	$88.46942\ (3.5-x)^3 + 142.4534\ (x-3)^3 + 220.8826\ (3.5-x) + 489.1054\ (x-3)$
[3.5,4]	$142.4534\ (4-x)^3 + 207.967\ (x-3.5)^3 + 489.1054\ (4-x) + 971.0082\ (x-3.5)$
[4,4.5]	$207.967\ (4.5-x)^3 + 315.6785\ (x-4)^3 + 971.0082\ (4.5-x) + 1764.862\ (x-4)$
[4.5,5]	$315.6785\ (5-x)^3 + 363.0691\ (x-4.5)^3 + 1764.862\ (5-x) + 3032.233\ (x-4.5)$
[5,5.5]	$363.0691\ (5.5-x)^3 + 744.5452\ (x-5)^3 + 3032.233\ (5.5-x) + 4844.207\ (x-5)$
[5.5,6]	$744.5452\ (6-x)^3 + 4844.207\ (6-x) + 7773\ (x-5.5)$

Table 3: Approximate values, exact values and errors of Ex 1 in [1, 1.5]

x	Using End - point – slope condition			Using Type I condition		
	Exact values of y	Approximate Values of y	Error	Exact Values of y	Approximate values of y	Error
1.05	3.226282	3.253731	-0.02745	3.226282	3.326946	-0.10066
1.1	3.51051	3.551175	-0.04066	3.51051	3.67101	-0.1605
1.15	3.861357	3.905576	-0.04422	3.861357	4.049307	-0.18795
1.2	4.28832	4.330181	-0.04186	4.28832	4.478954	-0.19063
1.25	4.801758	4.838232	-0.03647	4.801758	4.977069	-0.17531
1.3	5.41293	5.442974	-0.03004	5.41293	5.560769	-0.14784
1.35	6.134033	6.157653	-0.02362	6.134033	6.24717	-0.11314
1.4	6.97824	6.995512	-0.01727	6.97824	7.053389	-0.07515
1.45	7.959734	7.969796	-0.01006	7.959734	7.996544	-0.03681

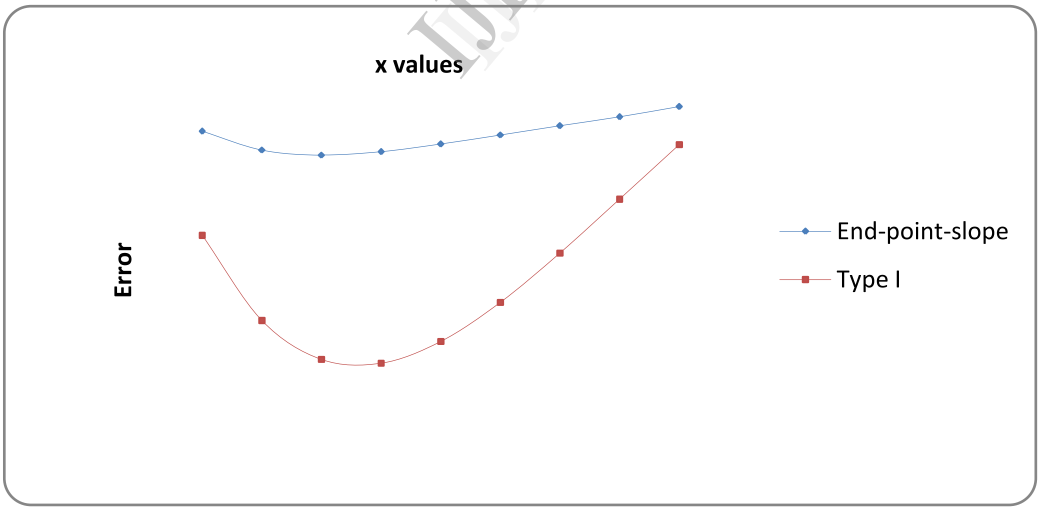

Fig.2: Comparison of errors for Ex.1

Table 4: Approximate values, exact values and errors of Ex. 1

x	Using End - point – slope condition			Using Type I condition		
	Exact values of y	Approximate Values of y	Error	Exact Values of y	Approximate values of y	Error
1.1	3.51051	3.551175	-0.04066	3.51051	3.67101	-0.1605
1.6	11.88576	11.85581	0.029946	11.88576	11.82435	0.061409
2.3	65.06343	65.02889	0.03454	65.06343	65.03072	0.032712
2.6	119.2138	119.1869	0.026858	119.2138	119.1943	0.019454
3.1	286.1915	286.1718	0.01969	286.1915	286.1362	0.055321
3.7	692.7396	692.6121	0.127504	692.7396	692.8865	-0.14693
4.3	1468.784	1468.987	-0.20247	1468.784	1467.694	1.090045
4.8	2546.24	2545.117	1.122665	2546.24	2549.941	-3.7012
5.2	3799.84	3802.817	-2.97649	3799.84	3788.541	11.29925
5.6	5504.718	5499.023	5.695105	5504.718	5525.268	-20.55

Table 5: Spline functions with end-point-slope conditions in the corresponding intervals

Interval	Cubic Spline function
[1,1.5]	$0.950728(1.5-x)^3+0.494674(x-1)^3+1.762318(1.5-x)+4.728514(x-1)$
[1.5,2]	$0.494674(2-x)^3+0.661355(x-1.5)^3+4.728514(2-x)+8.436721(x-1.5)$
[2,2.5]	$0.661355(2.5-x)^3+0.635674(x-2)^3+8.436721(2.5-x)+13.13696(x-2)$
[2.5,3]	$0.635674(3-x)^3+0.654117(x-2.5)^3+13.13696(3-x)+18.79071(x-2.5)$
[3,3.5]	$0.654117(3.5-x)^3+0.649982(x-3)^3+18.79071(3.5-x)+25.42564(x-3)$
[3.5,4]	$0.649982(4-x)^3+0.674316(x-3.5)^3+25.42564(4-x)+33.03554(x-3.5)$
[4,4.5]	$0.674316(4.5-x)^3+0.598039(x-4)^3+33.03554(4.5-x)+41.65692(x-4)$
[4.5,5]	$0.598039(5-x)^3+0.890368(x-4.5)^3+41.65692(5-x)+51.17535(x-4.5)$
[5,5.5]	$0.890368(5.5-x)^3-0.19443(x-5)^3+51.17535(5.5-x)+62.02933(x-5)$
[5.5,6]	$-0.19443(6-x)^3+3.858526(x-5.5)^3+62.02933(6-x)+72.59167(x-5.5)$

Table 6: Spline functions with Type I conditions in the corresponding intervals

Interval	Cubic spline function
[1,1.5]	$0.749447(x-1)^3+(1.5-x)+4.664821(x-1)$
[1.5,2]	$0.749447(2-x)^3+0.592994(x-1.5)^3+4.664821(2-x)+8.453811(x-1.5)$
[2,2.5]	$0.592994(2.5-x)^3+0.654347(x-2)^3+8.453811(2.5-x)+13.13229(x-2)$
[2.5,3]	$0.654347(3-x)^3+0.647788(x-2.5)^3+13.13229(3-x)+18.7923(x-2.5)$
[3,3.5]	$0.647788(3.5-x)^3+0.656625(x-3)^3+18.7923(3.5-x)+25.42398(x-3)$
[3.5,4]	$0.656625(4-x)^3+0.654074(x-3.5)^3+25.42398(4-x)+33.0406(x-3.5)$
[4,4.5]	$0.654074(4.5-x)^3+0.672363(x-4)^3+33.0406(4.5-x)+41.63833(x-4)$
[4.5,5]	$0.672363(5-x)^3+0.613313(x-4.5)^3+41.63833(5-x)+51.24461(x-4.5)$
[5,5.5]	$0.613313(5.5-x)^3-0.839463(x-5)^3+51.24461(5.5-x)+61.77086(x-5)$
[5.5,6]	$0.839463(6-x)^3+61.77086(6-x)+73.5563(x-5.5)$

Table 7: Approximate values, exact values and errors of Ex 2 in [1, 1.5]

x	Using end-point–slope conditions			Using Type I. conditions		
	Exact values of y	Approximate Values of y	Error	Exact Values of y	Approximate values of y	Error
1.05	1.123689	1.116166	0.007524	1.123689	1.133335	-0.00965
1.1	1.251393	1.23912	0.012273	1.251393	1.267232	-0.01584
1.15	1.383198	1.36852	0.014677	1.383198	1.402253	-0.01905
1.2	1.519181	1.504025	0.015156	1.519181	1.53896	-0.01978
1.25	1.65941	1.645292	0.014118	1.65941	1.677915	-0.01851
1.3	1.803943	1.79198	0.011964	1.803943	1.819681	-0.01574
1.35	1.952834	1.943745	0.009088	1.952834	1.96482	-0.01199
1.4	2.106128	2.100247	0.005881	2.106128	2.113893	-0.00776
1.45	2.263868	2.261143	0.002725	2.263868	2.267463	-0.00359

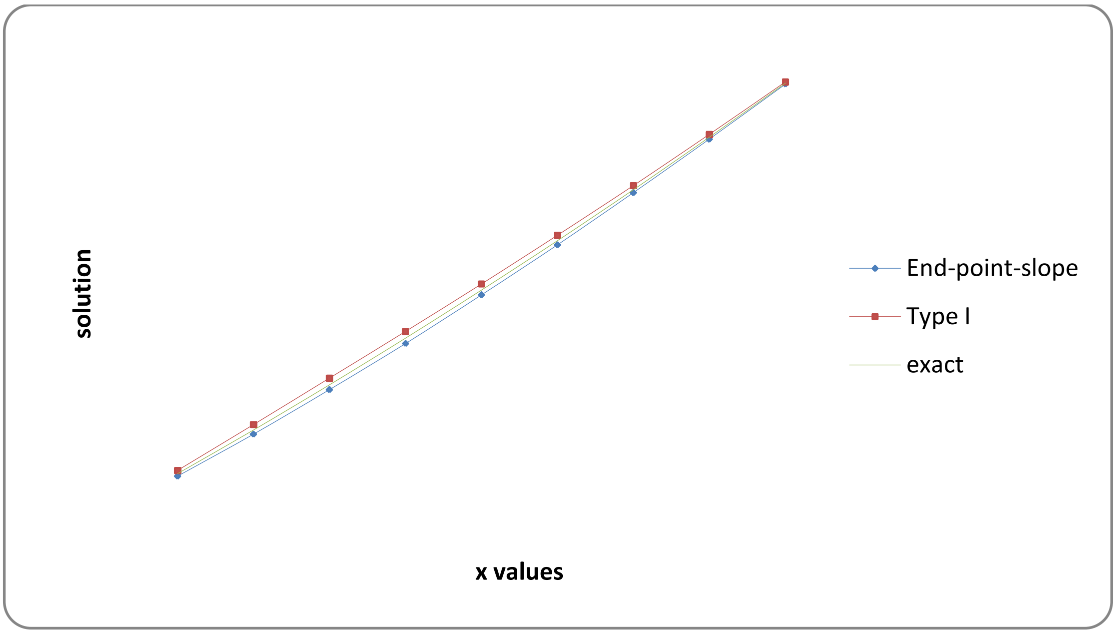

Fig.3: Comparison of approximate values and exact values for Ex.2

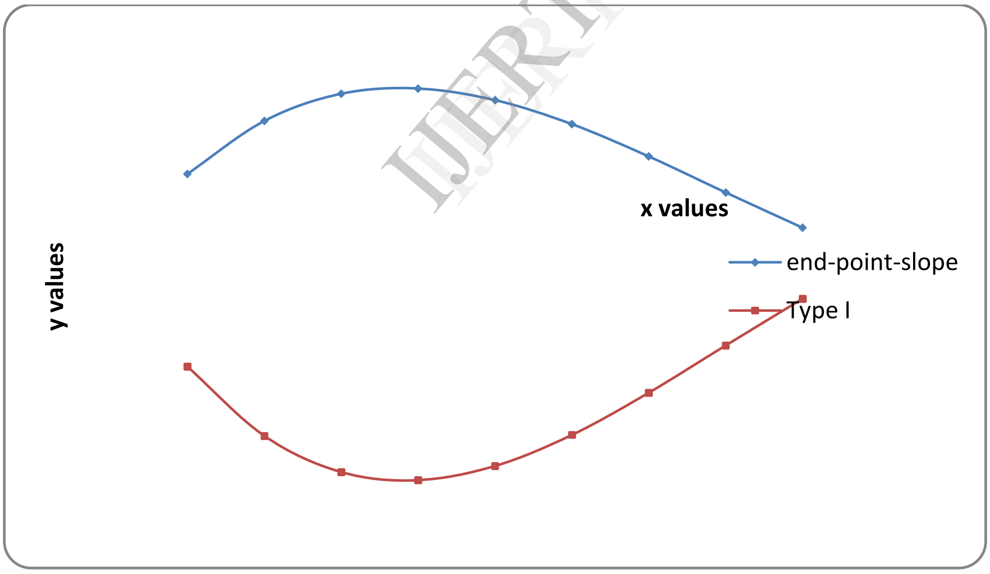

Fig.4: Error graph by two types of conditions at n=10 for Ex.2

Table 8: Approximate values, exact values and errors for Ex 2 in $[x_{i-1}, x_i]$, for $i = 1, 2 \ldots, 10$

x	Using End - point – slope condition			Using Type I condition		
	Exact values of y	Approximate Values of y	Error	Exact Values of y	Approximate values of y	Error
1.4	2.106128	2.100247	0.005881	2.106128	2.113893	-0.00776
1.9	3.888754	3.890361	-0.00161	3.888754	3.886708	0.002046
2.4	6.140211	6.139801	0.00041	6.140211	6.14077	-0.00056
2.8	8.287158	8.287353	-0.00019	8.287158	8.286872	0.000286
3.1	10.10136	10.10136	-1.2E-06	10.10136	10.10143	-7E-05
3.8	15.01978	15.0192	0.000587	15.01978	15.01989	-0.00011
4.1	17.42278	17.42366	-0.00088	17.42278	17.42261	0.000177
4.9	24.7002	24.69341	0.006784	24.7002	24.7016	-0.00141
5.1	26.71757	26.72986	-0.01229	26.71757	26.71502	0.002548
5.7	33.24587	33.15275	0.093122	33.24587	33.26518	-0.01931

Table 9: The absolute errors with two types of conditions

x	Ex 7.1		Ex 7. 2	
	End-point-slope condition	Type I condition	End-point-slope condition	Type I condition
1.2	0.04186	0.19063	0.015156	0.01978
1.25	0.03647	0.17531	0.014118	0.01851
1.3	0.03004	0.14784	0.011964	0.01574
1.35	0.02362	0.11314	0.009088	0.01199
1.4	0.01727	0.07515	0.005881	0.00776
1.45	0.01006	0.03681	0.002725	0.00359

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