

# $d^{dM}$ - Distance In Graphs

K. John Bosco

Assistant Professor, Department of Mathematics, St. Judes College Thoothoor,

Sathija J Prathisa S K

Research Scholar (Reg.No: 23113232092005),  
Department of Mathematics, St. Judes College  
Thoothoor,

Affiliated to Manonmaniam Sundaranar University,  
Abishekapatti, Tirunelveli – 627012, Tamilnadu, India.

## ABSTRACT

For two vertices  $u, v$  of a graph  $G$ , the usual distance  $d(u, v)$ , is the length of the shortest path between  $u$  and  $v$ . In this paper we introduced the concept of  $d^{dM}$ - distance by considering the degrees of various vertices presented in the path, in addition to the length of the path. We study some properties with this new distance. We define the eccentricities of vertices, radius and diameter of  $G$  with respect to the  $d^{dM}$ -distance. First we prove that the new distance is a metric on the set of vertices of  $G$ . We compare the usual, geodesic and  $d^{dM}$ - distances of two vertices  $u, v$  of  $V$ .

## KEYWORDS

Geodesic distance,  $d^{dM}$  - distance,  $d^{dM}$  - Eccentricity,  $d^{dM}$  - Radius and  $d^{dM}$  - Diameter.

## 1. INTRODUCTION

By a graph  $G$ , we mean a non-trivial finite undirected connected graph without multiple edges and loops. Following standard notations (for any unexplained notation and terminology we refer [2])  $V(G)$  or  $V$  is the vertex set of  $G$  and  $E(G)$  or  $E$  is the edge set of  $G = G(V, E)$ . Let  $u, v$  be two vertices of  $G$ . The standard or usual distance  $d(u, v)$  between  $u$  and  $v$  is the length of the shortest  $u - v$  path in  $G$ . Chartrand et al [3] introduced the concept of detour distance in graphs as follows: For two vertices  $u, v$  in a graph  $G$ , the detour distance  $D(u, v)$  is defined as the length of the longest  $u - v$  path in  $G$ . In this article we introduce a new distance, which we call as  $d^{dM}$  - distance between any two vertices of a graph  $G$ , and study some of its properties. This distance is significantly different from other distances. In some of the earlier distances, only path length was considered. Here we, in addition, consider the degree of  $u$  and  $v$  vertices present in a  $u - v$  path while defining its length. Using this length we define the  $d^{dM}$ -distance. Chartand et al introduced the concept of detour distance by considering the length of the longest path between  $u$  and  $v$ . Kathiresan et al [4] introduced the concept of superior distance and signal distance. Goldern Ebenezer et al [13]

introduced the concept of  $d^d$  - distance. In some of these distances only the lengths of various paths were considered. We introduce the concept of mean  $d^{dM}$ - distance in some graphs  $G$ .

## 2. $d^{dM}$ - DISTANCE IN GRAPHS

### DEFINITION 2.1

Let  $u, v$  be two vertices of a connected graph  $G$ . Then the mean  $d^{dM}$  - length of a  $u - v$  path defined as  $d^{dM}(u, v) = d(u, v) + \deg(u) + \deg(v) + \left\lceil \frac{\deg(u) + \deg(v)}{2} \right\rceil$  or  $\left\lfloor \frac{\deg(u) + \deg(v)}{2} \right\rfloor$ , where  $d(u, v)$  is the shortest distance between the vertices  $u$  and  $v$ . The  $d^{dM}$  - distance between two vertices  $u$  and  $v$  is defined as the  $d^{dM}$  - length of a  $u - v$  path.

### DEFINITION 2.2

The  $d^{dM}$  - eccentricity of any vertex  $v$ ,  $e^{d^{dM}}(v)$ , is defined as the maximum distance from  $v$  to any other vertex, i.e.,  $e^{d^{dM}}(v) = \max\{d^{dM}(u, v) : u, v \in V(G)\}$

### DEFINITION 2.3

Any vertex  $v$  for which  $d^{dM}(u, v) = e^{d^{dM}}(v)$  is called  $d^{dM}$  - eccentric vertex of  $v$ . Further, a vertex  $u$  is said to be  $d^{dM}$  - eccentric vertex of  $G$  if it is the  $d^{dM}$  - eccentric vertex of some vertex.

### DEFINITION 2.4

The  $d^{dM}$  - radius, denoted by  $r^{d^{dM}}(G)$ , is the minimum  $d^{dM}$  - eccentricity among all vertices of  $G$  i.e.,  $r^{d^{dM}}(G) = \min\{e^{d^{dM}}(v) : v \in V(G)\}$ . Similarly the  $d^{dM}$  - diameter,  $d^{dM}$

(G), is the maximum  $d^{dM}$  - eccentricity among all vertices of G.

**DEFINITION 2.5**

A graph is called  $d^{dM}$  self centered if radius is same as diameter, i.e.,  $r^{d^{dM}} = d^{d^{dM}}$ .

**THEOREM 2.6**

If G is any connected graph, then the  $d^{dpM}$  - distance is a metric on the set of vertices of G.

**PROOF:**

Let G be a connected graph and  $u, v \in V(G)$ . Then it is clear by definition that  $d^{dM}(u, v) \geq 0$  and  $d^{dM}(u, v) = 0$  which implies  $u = v$ . It is obvious that  $d^{dM}(u, v) = d^{dM}(v, u)$ . It can be proved that the distance  $d^{dM}$  satisfies the triangle inequality.

**Case (i):**

Let  $u, v, w \in V(G)$ . Let P and Q be  $u - w$  and  $w - v$  shortest paths of shortest lengths  $l(P)$  and  $l(Q)$  respectively in G. Then  $d^{dM}(u, w) = l(P) + \deg(u) + \deg(w) + \left\lceil \frac{\deg(u) + \deg(w)}{2} \right\rceil$  and  $d^{dM}(w, v) = l(Q) + \deg(w) + \deg(v) + \left\lceil \frac{\deg(w) + \deg(v)}{2} \right\rceil$ . Let  $R = PUQ$  be the  $u - v$  shortest path obtained by joining P and Q at w of length greater than one.

$$\begin{aligned} \text{Then } d^{dM}(u, w) + d^{dM}(w, v) &= [ l(P) + \deg(u) + \deg(w) + \left\lceil \frac{\deg(u) + \deg(w)}{2} \right\rceil ] + [ l(Q) + \deg(w) + \deg(v) + \left\lceil \frac{\deg(w) + \deg(v)}{2} \right\rceil ] \\ &= l(PUQ) + \deg(u) + \deg(v) + 2 \deg(w) + \left\lceil \frac{\deg(u) + \deg(w)}{2} \right\rceil + \left\lceil \frac{\deg(w) + \deg(v)}{2} \right\rceil \\ &= l(R) + \deg(u) + \deg(v) + 2 \deg(w) + \left\lceil \frac{\deg(u) + \deg(w)}{2} \right\rceil + \left\lceil \frac{\deg(w) + \deg(v)}{2} \right\rceil \\ &> l(R) + \deg(u) + \deg(v) + \left\lceil \frac{\deg(u) + \deg(w)}{2} \right\rceil + \left\lceil \frac{\deg(w) + \deg(v)}{2} \right\rceil \\ &= d^{dM}(u, v). \end{aligned}$$

Therefore  $d^{dM}(u, w) + d^{dM}(w, v) > d^{dM}(u, v)$ .

**Case (ii):**

Suppose  $R = PUQ$  be the  $u - v$  shortest path of length one. Then we take a vertex  $w$  other than  $u$  and  $v$  and so we have either  $u = w$  or  $v = w$ .

**Subcase (i):**

Suppose that  $u = w$ . Then  $d^{dM}(u, w) = 0$ .  
Therefore  $d^{dpM}(u, w) + d^{dpM}(w, v) = 0 + l(Q) + \deg(w) + \deg(v) + \left\lceil \frac{\deg(w) + \deg(v)}{2} \right\rceil$   
 $= l(PUQ) + \deg(u) + \deg(v) + \left\lceil \frac{\deg(u) + \deg(v)}{2} \right\rceil$ .  
 $d^{dM}(u, w) + d^{dM}(w, v) = d^{dM}(u, v)$ .

**Subcase (ii):**

Suppose that  $v = w$ . Proof is similar to subcase (i).

Thus the triangular inequalities hold and hence  $d^{dM}$  is a metric on the vertex set.

**COROLLARY 2.7**

For any three vertices  $u, v, w$  of a graph G,  $d^{dM}(u, v) \leq d^{dM}(u, w) + d^{dM}(w, v) - 2 \deg(w)$

**PROOF:**

Let G be a connected graph.

$$\begin{aligned} d^{dM}(u, w) + d^{dM}(w, v) &= l(PUQ) + \sum_{u, v = x \in PUQ} \deg(x) + \deg(w) + \sum_{u, v = x \in PUQ} \deg(x) + \deg(w) + \prod_{u, v = x \in PUQ} \deg(x) \\ &\geq d^{dM}(u, v) + 2 \deg(w). \end{aligned}$$

$$d^{dM}(u, w) + d^{dM}(w, v) \geq d^{dM}(u, v) + 2 \deg(w).$$

Hence  $d^{dM}(u, v) \leq d^{dM}(u, w) + d^{dM}(w, v) - 2 \deg(w)$ .

**PREPOSITION 2.8**

If G is a connected graph with two distinct vertices  $u, v$  are adjacent then  $d^{dM}(u, v) \leq 4(n - 1)$ .

**PROOF:**

Let  $u$  and  $v$  be two adjacent vertices. This implies that  $d(u, v) \leq n - 1$ . By definition,  $d^{dM}(u, v) = d(u, v) + \deg(u) + \deg(v) + \left\lceil \frac{\deg(u) + \deg(v)}{2} \right\rceil \leq \deg(u) + \deg(v) + (n - 1) + \left\lceil \frac{\deg(u) + \deg(v)}{2} \right\rceil$ . Since  $d(u, v) \leq n - 1$ ,  $\deg(v) \leq n - 1$  for any vertices  $u$  and  $v$ . Hence  $d^{dM}(u, v) \leq 4(n - 1)$ .

**THEOREM 2.9**

For every tree  $u, v \in T$ ,  $d^{dM}(u, v) > d(u, v)$ .

**PROOF:**

In a tree  $d^{dM}(u, v) = d(u, v)$ . We know that  $d^{dM}(u, v) = d(u, v) + \deg(u) + \deg(v) + \left\lceil \frac{\deg(u) + \deg(v)}{2} \right\rceil$

This implies that  $d^{dM}(u,v) > d(u,v)$ .

The following results are immediate for complete graph.

**THEOREM 2.10**

In path graph  $P_n$ , the product mean  $d^{dM}$  of any vertex,  $n \geq 3$  is given below

The diameter  $d^{dM} = n+3$ .

The radius  $r^{dM}$ , If  $n$  is odd then  $d^{dM}(u,v) = \left(\frac{n-1}{2}\right) + 5$ .

If  $n$  is even then  $d^{dM}(u,v) = \left(\frac{n}{2}\right) + 5$ .

**THEOREM 2.11**

(i) For a complete graph  $K_n$ ,  $d^{dM}(u,v) = 3n - 2$  for every  $u,v \in K_n$ .

(ii) For a complete bipartite graph  $G = K_{m,n}$ ,  $m = n$  and  $V_m \cup V_n = V(G)$ ,  $d^{dM}(u,v) = 3n + 2 \forall u,v \in V_m$ .

(iii) For a star graph  $G = K_{1,n}$ , if  $n$  is odd,  $d^{dM}(u,v) = 3\left(\frac{n+1}{2}\right) + 1 \forall v \in V(G)$

if  $n$  is even,  $d^{dM}(u,v) = 3\left(\frac{n}{2}\right) + 3 \forall v \in V(G)$

**PROOF:**

Above results are obvious by definitions.

**COROLLARY 2.12**

- (i) Complete graph  $K_n$  is self centered.
- (ii) Complete bipartite graph  $K_{m,n}$  ( $m = n$ ), is self centered.
- (iii) Star graph  $K_{(1,n)}$ , is self centered.

**OBSERVATION 2.13**

For any connected graph  $G$ , if  $u$  and  $v$  are end vertices of  $G$  and  $d(u,v) \leq n - 1$  then  $d^{dM}(u,v) \leq 4(n - 1)$ .

□

**CONCLUSION:**

Many researchers are concentrating various distance concepts in graphs. In this paper we have studied about  $d^{dM}$  - distance in graphs. Then we have defined  $d^{dM}$  - eccentricity,  $d^{dM}$  - radius and  $d^{dM}$  - diameter. Many results have been found.

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