Desargues Systems and a Model of a Laterally Commutative Heap in Desargues Affine Plane

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***Abstract* - In this paper we have obtained some property of heaps like algebraic structure with a ternary operation combined whith groupoids, by putting concept of ternary operation from multiplication, heap from multiplication and ternary groupoid. Also we show that in the same set, the existence of a laterally commutative heap, existence of a parallelogram space, existence of a narrowed Desargues system and existence of a subtractive grupoid are equivalent to each other. So, we constructed a model of a laterally commutative heap in Desargues affine plane.**

Key words: Heaps, ward groupoids, subtractive groupoids, parallelogram space, narrowed Desargues system, laterally commutative heap.

1. LATERALLY COMMUTATIVE HEAP AND SUBTRACTIVE GROUPID IN THE SAME SET

In the terminology of [1] we define the following definitions.

Definition 1.1. *Let* $[\ ]: B^3 \to B$ *be a ternary operation in a nonempty set B.* (*B*, [ ]) *is called heap if* $\forall a,b,c,d,e \in B$,

$$[[abc]de] = [ab[cde]] \quad (1)$$

$$[abb] = [bba] = a \quad (2)$$

Definition 1.2. Let $B^2 \to B$ be a binary operation in a set B, denoted whith $\cdot$ and called multiplication in *B*. Groupoid (*B*, $\cdot$) is called right transitive groupoid (shortly Ward groupoid) if

$$\forall a,b,c \in B,\ ac \cdot bc = ab. \quad (3)$$

Definition 1.3. *Ward groupoid* (*B*, $\cdot$) is called right solvable if $\forall a,b \in B$ *the equation* $ax = b$ *has a solution.*
Right solvable Ward groupoid is a quasigroup [3].
Lemma1.1. [2] *If* (*B*, $\cdot$) *is right solvable Ward groupoid,* than $\exists! u \in B$ *such that* $\forall a \in B$,

$$a \cdot a = u, \quad (4)$$

$$a \cdot u = a. \quad (5)$$

Consequence 1.1. *Right solvable Ward groupoid* (*B*, $\cdot$)

1. *has a unique right identity element and so is the element* $u = b \cdot b$ *, where b is an element of B;*
2. *the proposition hold*

$$ab = u \cdot ba,\ \forall a,b \in B. \quad (5')$$

Definition 1.4. *Let (B,* $\cdot$ *) be a multiplicative group and o be a fixed element in the set B. Ternary operation* [ ] in *B*, defined as

$$[abc] = ab \cdot oc,\ \forall a,b,c \in B, \quad (6)$$

is called ternary operation from multiplication by the element o, whereas heap (*B*, [ ]), in which *ternary operation* [ ] is defined from (6), is called *heap operation from multiplication by the element o.*

Proposition 1.1. *If* (*B*, $\cdot$) *is right solvable Ward groupoid and u is right identity element, than the structure* (*B*, [ ]), *in which* [ ] *is ternary operation from multiplication by u, is heap.*

In the terminology of [3] now we have this:
Definition 1.5. *Ward groupoid* (*B*, $\cdot$) *is called subtractive groupoid , if* $\forall a,b \in B$,

$$a \cdot ab = b. \quad (7)$$

According to this definition, it is obvious that $x = ab$ is a solution of the equation $ax = b$.

Proposition 1.2. *If the groupoid* $(B, \cdot)$ *is subtractive, than it is right solvable Ward groupoid.*

Proposition 1.3. *If the structure* $(B, [\ ])$ *is heap from multiplication by right identity element u of groupoid* $(B, \cdot)$, *which is subtractive, than hold the equation:*

$$[abc] = a \cdot bc\,,\ \forall a,b,c \in B. \tag{8}$$

Definition 1.6. *Let* $(B, [\ ])$ is a ternary structure and *o is a fixed element of the set B. Groupoid* $(B, \cdot)$ is called ternary groupoid according *o, if* $\forall a,b \in B$,

$$a \cdot b = [abo]. \tag{9}$$

Proposition 1.4. *If the structure* $(B, [\ ])$ *is heap, than its ternary groupoid* $(B, \cdot)$ *according to o is right solvable Ward groupoid* with a right identity element the given element $o \in B$.

Definition 1.7. *Heap* $(B, [\ ])$ *is called laterally commutative heap , if* $\forall a,b,c \in B$,

$$[abc]=[cba]. \tag{11}$$

Proposition1.5. *If the groupoid* $(B, \cdot)$ *is subtractive, than the structure* $(B, [\ ])$, in which the ternary operation [ ] *is defined from* (8), *is laterally commutative heap.*

Proposition 1.6. . *If the groupoid* $(B, \cdot)$ *is subtractive , than the structure* $(B, [\ ])$, *in which* [ ] is ternary operation of multiplication by right identity element of *B, is laterally commutative heap.*

Proposition 1.7. *If the structure* $(B, [\ ])$ *is commutative heap in lateral way, than its ternarygroupoid* $(B, \cdot)$ *according to the element o of B, is subtractive.*

Theorem 1.1. Existence of a right solvable Ward groupoids (B, ·) with right identity element *u* gives the existence of a heap (B, [ ]), exactly corrensponding heap from multiciplation according *u*; existence of a heap (B, [ ]) gives the existence of a right solvable Ward groupoids (B, ·), exactly of its ternarygroupoid of a given element.

Theorem 1.2. Existence of a substractive groupoid (B, ·) gives exsistence of a laterally commutative heap, exactly heap of multiciplation (B, [ ]) according to right identity element *u*; existence of a laterally commutative heap (B, [ ]) gives existence of a substractive groupoid, exactly ternargroupoid (B, ·) according to an element *o* in B.

Shortly, Theorem 1.2, shows that, the existence of a laterally commutative heap is equivalent to the existence of a subtractive groupoid on the same set.

2. DESARGUES SYSTEMS AND PARALLELOGRAM SPACE

Let *q be a quarternary relation in a nonempty set B,* respectively $q \subseteq B^4$. The fact that $(x, y, z, u) \in q$ we can denote as $q(x, y, z, u)$ for $(x, y, z, u) \in B^4$.

Definition 2.1. [4] *Pair (B, q), where q is a quarternary relation in B, is called Desargues system if the following propositions are true*:

$$D1.\ \forall x,y,a,b,c,d \in B,\ q(x,a,b,y) \wedge q(x,c,d,y) \Rightarrow q(c,a,b,d);$$

$$D2.\ \forall x,y,a,b,c,d \in B,\ q(b,a,x,y) \wedge q(d,c,x,y) \Rightarrow q(b,a,c,d);$$

$$D3.\ \forall (a,b,c) \in B^3,\ \exists! d \in B, q(a,b,c,d).$$

Lemma. 2.1. [5] *If (B, q) is Desargues system, than we have:*

$$1.\ \forall a,b \in B, q(a,a,b,b) \wedge q(a,b,b,a). \tag{12}$$

$$2.\ \forall a,b,c,d \in B,\ q(a,b,c,d) \Rightarrow q(b,a,d,c),\quad q(a,b,c,d) \Rightarrow q(d,c,b,a). \tag{13}$$

Theorem 2.1. [6] *Let B be a set in which is defined ternary operation* [ ] *and a quarternary relation q, such that the equivalence is valid*

$$[abc] = d \Leftrightarrow q(a,b,c,d),\ \forall a,b,c,d \in B. \tag{14}$$

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In these conditions, (B, q) *is Desargues system, if and only if* (B, [ ]) *is a heap.*

Definition 2.2. System (B, q) is called narrowed Desargues system *if it holds:*

D4. $\forall a,b,c,d \in B,\ q(a,b,c,d) \Rightarrow q(a,d,c,b).$

Theorem 2.2. *Let* B *be a set in which is defined ternary operation* [ ] *and a quartenary relation q, such that satisfy* (14).

In these conditions, (*B*, *q*) *is narrowed Desargues system, if and only if* (*B*, [ ]) *is laterally comutative heap.*

Definition 2.3. *Pair (B, p), where p is a quartenary relation in B, is called parallelogram space if the propositions are valid:*

P1. $\forall a,b,c,d \in B,\ p(a,b,c,d) \Rightarrow p(a,c,b,d);$

P2. $\forall a,b,c,d \in B,\ p(a,b,c,d) \Rightarrow p(c,d,a,b);$

P3. $\forall a,b,c,d,e,f \in B,\ p(a,b,c,d) \wedge p(c,d,e,f) \Rightarrow p(a,b,e,f);$

P4. $\forall (a,b,c) \in B^3, \exists! d \in B,\ p(a,b,c,d).$

Theorem 2.3. [6] *Let B be a set in which are defined quartenary operations p, q such that satisfy the equivalence*

$$q(a,b,c,d) \Leftrightarrow p(a,b,d,c),\ \forall a,b,c,d \in B. \tag{15}$$

In these conditions, (*B*, *p*) is a parallelogram space, *if and only if* (*B*, *q*) is narrowed *Desargues system*.

Theorem 2.4. *Let B be a set in which is defined ternary operation* [ ] and a quarternary relation *p, such that hold the relation*

$$[abc] = d \Leftrightarrow p(a,b,d,c),\ \forall a,b,c,d \in B. \tag{16}$$

In these conditions, (*B*, *p*) is parallelogram space *if and only if* (*B*, [ ]) *is a laterally commutative heap.*

Theorem 2.5. *Let B be a set in which is defined multiplication* $\cdot$ *and a quarternay relation q, such that equivalence is valid*

$$a \cdot bc = d \Leftrightarrow q(a,b,c,d),\ \forall a,b,c,d \in B. \tag{17}$$

In these conditions, (*B*, $\cdot$) is a substractive *groupoid, if and only if* (*B*, *q*) *is narrowed Desargues system*.

Consequence 2.1. *Let B be a set in which is defined multiplication* $\cdot$ *and a quarternay relation q, such that equivalence is valid*

$$a \cdot bc = d \Leftrightarrow p(a,b,d,c),\ \forall a,b,c,d \in B.$$

In these conditions, (*B*, $\cdot$) *is subtractive groupoid, if and only if* (*B*, *p*) *is parallelogram space.*

From Theorems 2.2, 2.3, 2.4 dhe 2.5, it is obviously that:

Theorem 2.6. In the same set, the existence of a laterally commutative heap, existence of a parallelogram space, existence of a narrowed Desargues system and existence of a subtractive grupoid are equivalent to each other.
Following proposition gives sufficient condition of existence of Desargues system.

Proposition 2.1 *Let B be a set in which is defined multiplication* $\cdot$ *and a quarternay relation q, such that equivalence is valid*

$$ab = cd \Leftrightarrow q(a,b,d,c),\ \forall a,b,c,d \in B. \tag{18}$$

In these conditions,

1. *if u is right identity element in* (*B*, $\cdot$), *than hold the equivalence*

$$ab = c \Leftrightarrow q(a,b,u,c),\ \forall a,b,c \in B.$$

2. *if* (*B*, $\cdot$) *is Ward quasigroup, than* (*B*, *q*) *is Desargues system.*

Consequence 2.2. *Let B be a set in which is defined multiplication* $\cdot$ *and a quarternay relation q, such that equivalence* (18) *hold.*

In these conditions, if (*B*, $\cdot$) *is subtractive groupoid, than* (*B*, *q*) *is Desargues system*.

3. Model of a laterally commutative heap in Desargues affine plane

Let incidence structure A=(Π, Λ, I) be an *Desargues affine plane.*

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In Desargues affine plane vector is defined like an ordered pair of points from Π. If this point is a pair (A,B) of distinct point A,B and we denote $\overrightarrow{AB}$. $\overrightarrow{AB}$ is called *zero vector* if $A=B$.

Equality of vectors is defined:

(i) $\overrightarrow{AA}=\overrightarrow{DC}\Leftrightarrow D=C$;

(ii) $\forall\overrightarrow{AB}$, $\overrightarrow{AB}=\overrightarrow{AB}$;

(iii) If a direction lines of two nonzero vectors $\overrightarrow{AB}$, $\overrightarrow{DC}$ are distinct lines then, $\overrightarrow{AB}=\overrightarrow{DC}\Leftrightarrow AB\,||\,DC$ and $AD\,||\,BC$ (Fig. 1).

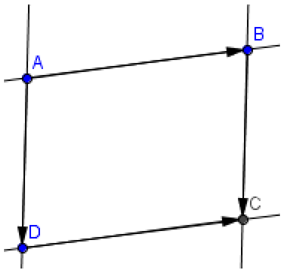

Fig. 1

(iv) If a direction lines of two nonzero vectors $\overrightarrow{AB}$, $\overrightarrow{DC}$ are the same lines, then $\overrightarrow{AB}=\overrightarrow{DC}$, when there exixsts a vector $\overrightarrow{MN}$ with direction line $MN\neq AB$ such that $\overrightarrow{AB}=\overrightarrow{MN}$ and $\overrightarrow{MN}=\overrightarrow{DC}$ (Fig. 2).

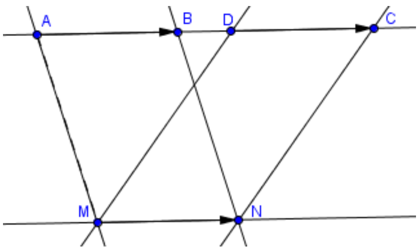

Fig. 2

From this definition hold

$$\overrightarrow{AB}=\overrightarrow{DC}\Leftrightarrow\overrightarrow{AD}=\overrightarrow{BC} \qquad (19)$$

When the points A,B,C are nocolinear, the point D, by (iii), are the four vertex of a parallelogram $ABCD$. (Fig. 1).

When the points A,B,C are collinear, we have the following cases for determine the point D:

1. $A=B=C$. In this case from $\overrightarrow{AB}=\overrightarrow{DC}$ we have $\overrightarrow{AA}=\overrightarrow{DA}$; by ($i$) $D=A$.
2. $A=B\neq C$. In this case from $\overrightarrow{AB}=\overrightarrow{DC}$ we have $\overrightarrow{AA}=\overrightarrow{DC}$; by ($i$) $D=C$.
3. $A\neq B=C$. In this case from $\overrightarrow{AB}=\overrightarrow{DC}$ we have $\overrightarrow{AB}=\overrightarrow{DB}$; by ($ii$) $D=A$.
4. $B\neq A=C$. In this case from $\overrightarrow{AB}=\overrightarrow{DC}$ we have $\overrightarrow{AB}=\overrightarrow{DA}$;by ($iv$), the point D is determine from two paralelograms;
5. A,B,C are distict. In this case by (iv), the point D is determined whith two paralelograms.

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So, $\forall (A,B,C) \in \Pi^3$, $\exists! D \in \Pi$, $\overrightarrow{AB} = \overrightarrow{DC}$. Let we determine in Π ternary operation [ ]: $\Pi^3 \rightarrow \Pi$ such that:

$$[ABC] = D \Leftrightarrow \overrightarrow{AB} = \overrightarrow{DC},\ \forall (A,B,C) \in \Pi^3 \qquad (20)$$

In this way we constructed ternary structure (Π, [ ]) in an Desargues affine plane. In the following proposition we prove that this is the model of an laterally commutative heap in such plane.

Proposition 3.1. Ternary structure (Π, [ ]) is a laterally commutative heap.
Proof. Let $A,B,C,D,E \in \Pi$. We denote $[ABC] = X$, $[XDE] = Y$, $[CDE] = Z$, $[ABZ] = T$. By (19) and (20) we have:

$$\left.\begin{array}{l} \left.\begin{array}{l} [ABC]=X \Leftrightarrow \overrightarrow{AB} = \overrightarrow{XC}; \\ [XDE]=Y \Leftrightarrow \overrightarrow{XD} = \overrightarrow{YE} \overset{(19)}{\Leftrightarrow} \overrightarrow{XY} = \overrightarrow{DE} \end{array}\right]; \\ \left.\begin{array}{l} [CDE]=Z \Leftrightarrow \overrightarrow{CD} = \overrightarrow{ZE} \overset{(19)}{\Leftrightarrow} \overrightarrow{CZ} = \overrightarrow{DE}; \\ [ABZ]=T \Leftrightarrow \overrightarrow{AB} = \overrightarrow{TZ} \end{array}\right] \end{array}\right] \Rightarrow \left.\begin{array}{l} \overrightarrow{AB} = \overrightarrow{TZ}; \\ \overrightarrow{XY} = \overrightarrow{CZ} \overset{(19)}{\Rightarrow} \overrightarrow{XC} = \overrightarrow{YZ} \end{array}\right]$$

Hence, $\overrightarrow{YZ} = \overrightarrow{TZ} \Rightarrow Y = T \Rightarrow [[ABC]DE] = [AB[CDE]]$. So,

$$[[ABC]DE] = [AB[CDE]],\ \forall A,B,C,D,E \in \Pi \qquad (*)$$

In Fig. 3 we illustrate this.

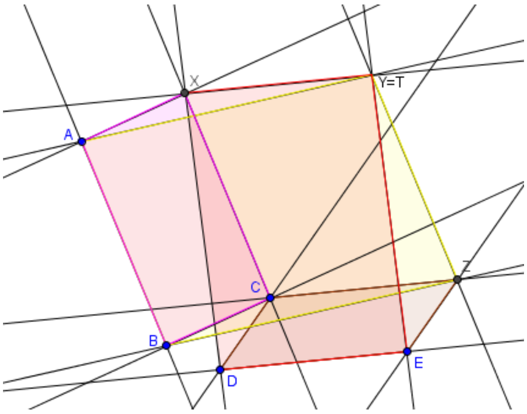

Fig. 3

Hence, by (20), $[ABB] = D \Leftrightarrow \overrightarrow{AB} = \overrightarrow{DB}$. By (*ii*), $\overrightarrow{AB} = \overrightarrow{DB} \Rightarrow D = A$. This implicate $[ABB] = A$. Also by (20), $[BBA] = D \Leftrightarrow \overrightarrow{BB} = \overrightarrow{DA}$. By (*i*), $\overrightarrow{BB} = \overrightarrow{DA} \Rightarrow D = A$. This implicate $[BBA] = A$. So hold,

$$[ABB] = [BBA] = A \qquad (**)$$

Finaly, for each three points A,B,C from Π, we have

$$\left.\begin{array}{l} [ABC] = D \Leftrightarrow \overrightarrow{AB} = \overrightarrow{DC}; \\ [CBA]=E \Leftrightarrow \overrightarrow{CB} = \overrightarrow{EA} \Leftrightarrow \overrightarrow{EA} = \overrightarrow{CB} \overset{(19)}{\Leftrightarrow} \overrightarrow{EC} = \overrightarrow{AB} \end{array}\right] \Rightarrow \overrightarrow{EC} = \overrightarrow{DC} \Rightarrow D = E.$$

So, we have

$$[ABC] = [CBA],\ \forall (A,B,C) \in \Pi^3 \qquad (***)$$

The results (*), (**), (***), by the Definition 1.1. and Definition 1.7, shows that this proposition hold.

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