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Determination of Expected Busy Periods in Faster and Slower Arrival Rates of an Interdependent M/M/1 :(∞; GD) Queueing Model with Controllable Arrival Rates

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Abstract

In the present paper, an interdependent *M/M/1 :(∞; GD)* queueing model with controllable arrival rates has been taken into our consideration with an objective to determine the expected busy periods in both cases of faster and slower arrival rates taking into account that both the arrival and service processes are interdependent and these discrete random variables follow a bivariate Poisson distribution. By the end of the present paper, a special attention has also been made in our conclusion in order to focus the significance of the investigated average busy periods.

Key Words: Interdependent queueing model, bivariate Poisson process, controllable arrival rates, busy period, probability generating function and Laplace transform etc.
AMS Subject Classification (2000): 62 E 15, 62 H 10

1. Introduction

Wastage of time either of customer or server due to waiting, reneging and balking is a common phenomenon in the study of queueing systems. Both states of the idle server and waiting customer in a queueing system are the basic goals to be reduced so that an optimum balance of the queueing system may be achieved. This is the cause why the busy period analysis of a realistic queueing model has attracted the keen interest of researchers of recent times. The busy period analysis of a queueing system has become an important issue for recent researchers in the literature because it plays a vital role in the study of queueing problems for forecasting the behaviour of the queueing systems. In general, most of the previous researchers [1, 6, 7, 8] and references therein have contributed to analyze the busy period of a variety of queueing models considering the arrival and service processes as independent but there are some real queueing situations having interdependent arrival and service processes. Moreover, the arrival rate of queueing system may be controlled in order to reduce the queue length. Queueing models with controllable arrival rates have been studied by a few noteworthy researchers [3, 4, 6 & 10] which reveals the fact that there is still an utmost demand of analyzing an interdependent queueing models with controllable arrival rates. Srinivasa Rao et al [3] have confined to obtain some useful performance measures of an $M/M/1/\infty$ interdependent queueing model with controllable arrival rates under steady state conditions such as the average system size and average waiting time. Of late, Pal [6] has focused his attention pertaining to establish the cost per unit time of the same interdependent queueing model studied by Srinivasan Rao et al [3] with its version of finite waiting space. Thereafter, Thiagarajan and Srinivasan [4] contributed to analyze the $M/M/1/\infty$ interdependent queueing model with controllable arrival rates and bulk arrivals. Recently, Maurya [6, 10] succeeded to investigate the probability generating functions of an interdependent *M/M/1:(∞;GD)* queueing model with bivariate Poisson process and controllable arrival rates. In this sequential work, we consider here the same interdependent *M/M/1:(∞;GD)* queueing model studied by Maurya [10] under the same postulates in order to further investigate the average busy periods in two different arrival rates of faster and slower.

2. Description and Postulates of the Model

In our current study, we consider an interdependent *M/M/1 :(∞; GD)* queueing model with bivariate Poisson process and controllable arrival rates studied by Maurya [10] to further obtain the average busy periods. The arrival pattern of customers are controlled by the system that it allows two arrival rates λ_1 and λ_2; $\lambda_1 > \lambda_2$. Without loss of generality we assume that whenever the system size attains a fixed number S, the arrival rate reduces form λ_1 to λ_2 and the arrival rate λ_2 remains unchanged till the system size is greater than R; $0 \le R < S$. But as soon as the system size reduces to R, the arrival rate λ_2 changes back to λ_1 and the same pattern of change of arrival rates is repeated during the complete busy period of the system. Moreover, we assume that *{X(t)}* and *{Y(t)}* representing the arrival and service processes respectively are interdependent and these discrete random variables follow a bivariate Poisson distribution as presumed earlier by Maurya [6, 10] with their probability mass function as given below:

$$\mathrm{P}\{(X(t)=x, Y(t)=y)\} = \frac{e^{-(\lambda_2+\mu-\nu)t}\sum_{k=0}^{\min(\lambda i,\mu)}(x,y)(\nu t)^k[(\lambda_i-\nu)t]^{x-k}[(\mu-\nu)t]^{y-k}}{k!\,(\mathrm{x}-\mathrm{k})!\,(\mathrm{y}-\mathrm{k})!}$$

$$x, y = 1,2,3\ldots\ldots;\ \lambda_i > 0;\ i = 1,2 \text{ and } \mu > 0;\ 0 < \boldsymbol{\nu} < \min(\lambda_I, \mu). \qquad (2.1)$$

Here μ is the mean service rate and $\boldsymbol{\nu}$ is the covariance between arrival and service processes.

In addition to this, we have underlying postulates for the purpose of our analysis:

Postulate 2.1: The probability that there is one arrival and no service completion during a small interval of time Δt is $(\lambda_i - \boldsymbol{\nu})\ \Delta t + O(\Delta t)$, when the system has arrival rate λ_i, i = 1, 2.

Postulate 2.2: The probability that there is neither arrival and nor service completion during a small interval of time Δt is $1 - (\lambda_i + \mu - 2\boldsymbol{\nu})\ \Delta t + O(\Delta t)$, when the system has arrival rate λ_i, i = 1, 2.

Postulate 2.3: The probability that there is no arrival and one service completion during a small interval of time Δt is $(\mu - \boldsymbol{\nu})\ \Delta t + O(\Delta t)$, whatever be the arrival rate λ_i.

Postulate 2.4: The probability that there is one arrival and one service completion during a small interval of time Δt is $\boldsymbol{\nu}\Delta t + O(\Delta t)$, whatever is the arrival rate λ_i, i = 1, 2.

Now we define following probability generating functions:

$$\mathrm{H}(z,t_1) = \sum_{k=0}^{S-1} P_k(t_1)\, z^k \qquad (2.2)$$

$$\mathrm{W}(z,t_2) = \sum_{k=R+1}^{\infty} P_k(t_2)\, z^k \qquad (2.3)$$

and we use symbols $\hat{\mathrm{H}}(z, s_1)$ and $\hat{\mathrm{W}}(z, s_2)$ for the Laplace transform of $\mathrm{H}(z, t_1)$ and $\mathrm{W}(z, t_2)$ respectively in following equations:

$$\hat{\mathrm{H}}(z,s_1) = \int_0^\infty e^{-st}\,\mathrm{H}(z,t_1)dt\,; \quad |z| \le 1 \qquad (2.4)$$

$$\hat{W}(z,s_2) = \int_0^\infty e^{-st} W(z,t_2)dt \ ; |z| \le 1 \tag{2.5}$$

4. Determination of Busy Periods

Case I: Busy Period for Faster Arrival Rate

In view of Maurya [6, 10], we have $\hat{H}(z,s_1)$ as following:

$$\hat{H}(z,s_1) = \frac{z^2-(1-z)[(\mu-\nu)-(\lambda_1-\nu)z][\{z_1(0)\}^2+(\mu-\nu)\{z_1(0)\}^{R+1}\ \hat{P}_{R+1}(s_2)]}{(\lambda_1-\nu)\{z-z_1(0)\}\{z_2(0)-z\}}$$

$$+\frac{(\mu-\nu)z^{R+1}\hat{P}_{R+1}(s_2)}{(\lambda_1-\nu)\{z-z_1(0)\}\{z_2(0)-z\}} \tag{4.1}$$

And the $\hat{H}(z,s_1)$ converges in the region of the unit circle; $|z| \le 1$ and Re $(s_1) > 0$.

On making use of property of $\hat{P}_0(s_1) = \hat{H}(0, s_1)$ in equation (4.1), one may readily get $\hat{P}_0(s_1)$ as following:

$$\hat{P}_0(s_1) = \frac{(\mu-\nu)}{(\lambda_1-\nu)s_1 z_2(0)} + \frac{\hat{P}_{R+1}(s_2)(\mu-\nu)^{R+1}}{(\lambda_1-\nu)^R\ s_1\{z_2(0)\}^R} \tag{4.2}$$

From equation (4.2), one can easily obtain

$$-\frac{d}{ds_1}[s_1\hat{P}_0(s_1)]_{s_1=0}$$

$$= \frac{1}{(\mu-\lambda_1)} - (\mu-\nu)[\lim_{s_1=0}\frac{d}{ds_1}\hat{P}_{R+1}(s_2) - \frac{R}{(\mu-\lambda_1)}\lim_{s_1=0}\frac{d}{ds1}\hat{P}_{R+1}(s_2)] \tag{4.3}$$

In the light of Gross and Harris [2], equation (4.3) yields after a little simplification as following:

$$\frac{d}{ds_1}[s_1\hat{P}_0(s_1)]_{s_1=0} = \frac{d}{ds_1}[\hat{P}_0(s_1)]_{s_1=0}$$

$$= -\int_0^\infty [t_1\hat{P}_0(t_1)dt_1 = E[T_1] \tag{4.4}$$

Combining equations (4.3) and (4.4), we readily get $E[T_1]$ as following

$$E[T_1] = \frac{1}{(\mu-\lambda_1)} - (\mu-\nu)[\lim_{s_1=0}\frac{d}{ds_1}\hat{P}_{R+1}(s_2) - \frac{R}{(\mu-\lambda_1)}[\lim_{s_1=0}\hat{P}_{R+1}(s_2)] \tag{4.5}$$

Particularly, when $\nu = 0$ and $\lambda_1 = \lambda$, the average length of busy period of the conventional model discussed in Gross and Harris [2] may be obtained easily from equation (4.5).
Finally, in the light of equations (4.3) and (4.5), we may have $E[T_1]$, the expected busy period in faster arrival rate as following:

$$E[T_1] = \frac{(1-R)}{(\mu-\lambda)} + (\mu-\nu)[\frac{(R-1)}{(\mu-\lambda)} - 1] \tag{4.6}$$

Case II: Busy Period for Slower Arrival Rate

In view of Maurya [6, 10], we may have $\hat{W}(z,s_2)$ as following:

$$\hat{W}(z,s_2) = \frac{z^{R+2}-(1-z)(\mu-\nu)z\,[\{z_1(1)\}^{R+2}(\lambda_1-\nu)\hat{P}_{S-1}(s_1)\{z_1(1)\}^{S+1}+(\lambda_1-\nu\hat{P}_{S-1}(s_1)\{z_1(1)\}^{S+1}]}{(\lambda_2-\nu)\{z-z_1(1)\}\{z_2(1)-z\}} \tag{4.7}$$

From equation (4.7), one can readily get

$$s_2\hat{W}(0,s_2) = \frac{(\mu-\nu)}{(\lambda_2-\nu)\{z_2(1)\}^{R+1}} + \frac{(\mu-\nu)^S(\lambda_1-\nu)\hat{P}_{S-1}(s_1)}{(\lambda_2-\nu)^S\{z_2(1)\}^S} \tag{4.8}$$

In view of Gross and Harris [2] and equation (4.8), it is easy to obtain

$$\frac{d}{ds_2}[s_2\hat{W}(z,s_2)]_{s_2=0} = \frac{d}{ds_2}[s_2\hat{W}(0,s_2)\;]_{s_2=0}$$

$$= -\int_0^\infty [t_2\hat{P}_{R+1}(t_2)dt_2 = E\,[T_2] \tag{4.9}$$

Thus by virtue of equations (4.8) and (4.9), we may get $E\,[T_2]$ as in the following equation

$$E\,[T_2] = \frac{1}{(\mu-\lambda_2)} - (\lambda_1-\nu)[\frac{S\lim_{s_2=0}\hat{P}_{S-1}(s_1)}{(\mu-\lambda_2)} - \lim_{s_2=0}\frac{d}{ds_2}\hat{P}_{S-1}(s_1)] \tag{4.10}$$

It is interesting to remark here that when $\nu = 0$ and $\lambda_2 = \lambda$, the equation (4.10) agrees to the corresponding result of the conventional model discussed already in Gross and Harris [2]. Finally, in view of equations (4.8) and (4.10), we may have $E\,[T_2]$, the average length of the busy period when the system has slower rate of arrivals as following:

$$E\,[T_2] = \frac{1}{(\mu-\lambda_2)} + \frac{1}{(\mu-\lambda_2)}[\frac{(\lambda_2-\nu)^R(\lambda_2-\nu)S}{(\mu-\nu)^R 2(\lambda_1-\nu)} + R + 1] \tag{4.11}$$

5. Conclusion

The busy period analysis of a queueing model is the most important performance measure in study of queueing problems because it plays a key role for determining cost per unit time of the model. Before proceeding to obtain the cost per unit time, the average busy period of server in the system must be known. In the present paper, we have succeeded to establish the average busy periods for two different cases of slower and faster arrival rates of an interdependent *M/M/1/∞:(∞ ;GD)* queueing model with controllable arrival rates taking into consideration that the arrival and service processes follow bivariate Poisson distribution. Therefore, the average busy periods investigated in the current study for the queueing model taken into our consideration reveal the practical significance. Moreover, two particular cases in states of both slower and faster arrival rates have also been discussed to highlight the conformity of the corresponding result of the conventional model discussed already in Gross and Harris [2] on taking into account that arrival and service rates are uncorrelated ($\nu = 0$).

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