Double Laplace Transform & It's Applications

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Abstract: In this paper, we applied the method of Double Laplace Transform for solving the Partial Differential Equations, that is, one dimensional Wave & Heat equation. Through this methodology we tried to prove that this method is very effective & convenient for solving Partial Differential Equation. The scheme is tested through some examples & the results demonstrate reliability.

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1. Introduction

Integral Transform [1, 2] is one of the most known methods to solve partial differential equations. The Wave equation & Heat equation as the fundamental equations in mathematical Physics & occur in many branches of Physics, in Applied mathematics as well as in Engineering. Eltayeb and Kilicman [3] have applied the double Laplace transform to solve general linear telegraph and partial integro-differential equations.

In 2011 [4], Aghilli & Moghaddam proved certain Theorems on Two Dimensional Laplace Transform & applied on Non-Homogeneous parabolic Partial differential equations.

Recently in 2013 [5], R. R. Dhunde has discussed & proved different properties of Double Laplace Transform.

In this study, we use the Double Laplace Transform to solve a first & second order Partial Differential equation, specially one dimensional Wave & Heat equation. Through this method the Partial Differential equation is solved without converting it into Ordinary Differential equation, therefore no need to find complete solution of Ordinary Differential equation. This is the biggest advantage of this method. Therefore Double Laplace Transform technique is very convenient & effective.

The scheme is tested through three different examples which are being referred from

[6, 7].

Definition of double Laplace transform:

First of all, we recall the following definitions given by Estrin & Higgins [2].

Let *f(x, t)* be a function of two variables *x* and *t*, where *x*, *t* > 0. The double Laplace transform of *f(x, t)* is defined as

$$L_t L_x \{f(x,t)\} = \overline{f}(p,s) = \int_0^\infty e^{-st} \int_0^\infty e^{-px} f(x,t)\,dx\,dt \tag{1}$$

whenever the improper integral converges. Here p, s are complex numbers.

Existence of double Laplace transforms:

Let f(x, t) be a continuous function on the interval [0, ∞) which is of exponential order, that is, for some a, bϵ R.

Consider $\quad \mathrm{Sup}_{x>0,t>0} \frac{|f(x,t)|}{e^{ax+bt}} < \infty$

In this case, the double Laplace transform of f(x, t) that is

$$L_t L_x \{f(x,t)\} = \overline{f}(p,s) = \int_0^\infty e^{-st} \int_0^\infty e^{-px} f(x,t)\,dx\,dt$$

exists for all $p > a$ & $s > b$ & is in fact infinitely differentiable with respect to $p > a$ & $s > b$.

All functions in this study are assumed to be of exponential order.

2. Double Laplace Transforms of Partial Derivatives:

Double Laplace Transform for first partial derivatives with respect to x is defined as follows:

$$L_t L_x \{f_x(x, t)\} = p \bar{f}(p, s) - \bar{f}(0, s) \qquad (2)$$

Similarly, Double Laplace Transform for first partial derivatives with respect to t is given by

$$L_t L_x \{f_t(x, t)\} = s \bar{f}(p, s) - \bar{f}(p, 0) \qquad (3)$$

Double Laplace Transform for second partial derivatives with respect to x is defined by

$$L_t L_x \{f_{xx}(x, t)\} = = p^2 \bar{f}(p, s) - p \bar{f}(0, s) - \bar{f}_x(0, s) \qquad (4)$$

In a similar manner, Double Laplace Transform for second partial derivatives with respect to t can be deduced from a single Laplace Transform

$$L_t L_x \{f_{tt}(x, t)\} = s^2 \bar{f}(p, s) - s \bar{f}(p, 0) - \bar{f}_t(p, 0) \qquad (5)$$

3. Applications of Double Laplace Transform:

3.1 Double Laplace Transform & First order Partial Differential Equation

Example: Find the bounded solution of $u_x = 2u_t + u$, $u(x, o) = e^{-3x}$ for $x > 0, t > 0$.

Solution: Taking the Double Laplace Transform, we obtain

$$L_t L_x \{u_x\} = L_t L_x \{2u_t + u\}$$

$$\Rightarrow p \bar{u}(p, s) - \bar{u}(0, s) = 2 \{s \bar{u}(p, s) - \bar{u}(p, 0)\} + \bar{u}(p, s)$$

$$\Rightarrow (p - 2s - 1) \bar{u}(p, s) = \bar{u}(0, s) - 2 \bar{u}(p, 0)$$

But $u(x, 0) = e^{-3x} \Rightarrow \bar{u}(p, 0) = \frac{1}{p+3}$

$$\Rightarrow \bar{u}(p, s) = \frac{1}{(p - 2s - 1)} \bar{u}(0, s) - \frac{2}{(p - 2s - 1)(p+3)}$$

$$\Rightarrow \bar{u}(p, s) = \frac{1}{(p - 2s - 1)} \bar{u}(0, s) - \frac{1}{s+2} \left\{\frac{1}{p-2s-1} - \frac{1}{p+3}\right\}$$

Using L_x^{-1}, we get

$\Longrightarrow \bar{u}(\mathrm{x}, \mathrm{s}) = e^{(2s+1)x}\ \bar{u}(0, s) - \frac{1}{s+2}\ e^{(2s+1)x} + \frac{1}{s+2}\ e^{-3x}$

$$\Longrightarrow \bar{u}(\mathrm{x}, \mathrm{s}) = e^{(2s+1)x}\left\{\bar{u}(0, s) - \frac{1}{s+2}\right\} + \frac{1}{s+2}\ e^{-3x} \qquad (6)$$

Now u(x, t) is bounded as x → ∞ & hence $\bar{u}$(x, s) is bounded as x → ∞.

Hence from (6), $\bar{u}(0, s) - \frac{1}{s+2} = 0$

$\Longrightarrow \bar{u}(0, s) = \frac{1}{s+2}$

Therefore, $\bar{u}(\mathrm{x}, \mathrm{s}) = \frac{1}{s+2}\ e^{-3x}$

By Inverse Laplace Transform, we get bounded solution

$\mathrm{u}(\mathrm{x}, \mathrm{t}) = \ e^{-3x}\ e^{-2t}$.

3.2 Double Laplace Transform & One Dimensional Heat Equation

Example: Solve $u_t = \mathrm{k}\ u_{xx}$, u(x, 0) = sinπx , u(0, t) = 0 , u(1, t) = 0, 0$< x < 1, t > 0$.

Solution: $\mathrm{L}_{\mathrm{t}}\mathrm{L}_{\mathrm{x}}\{u_t\} = \mathrm{k}\ \mathrm{L}_{\mathrm{t}}\mathrm{L}_{\mathrm{x}}\{u_{xx}\}$

$\Longrightarrow \mathrm{s}\ \bar{u}(\mathrm{p}, \mathrm{s}) - \bar{u}(p, 0) = \mathrm{k}\ \{p^2\bar{u}(\mathrm{p}, \mathrm{s}) - \mathrm{p}\ \bar{u}(0, \mathrm{s}) - \bar{u}_x(0, \mathrm{s})\}$

But u(0, t) = 0 $\Longrightarrow \bar{u}(0, \mathrm{s}) = 0$ & u(x, 0) = sin$\pi x \Longrightarrow \bar{u}(\mathrm{p}, 0) = \frac{\pi}{\mathrm{p}^2+\pi^2}$

$\Longrightarrow \mathrm{s}\ \bar{u}(\mathrm{p}, \mathrm{s}) - \frac{\pi}{\mathrm{p}^2+\pi^2} = \mathrm{k}\ \{p^2\bar{u}(\mathrm{p}, \mathrm{s}) - \bar{u}_x(0, \mathrm{s})\}$

$\Longrightarrow (\mathrm{k}\ p^2 - \mathrm{s})\ \bar{u}(\mathrm{p}, \mathrm{s}) = k\bar{u}_x(0, \mathrm{s}) - \frac{\pi}{\mathrm{p}^2+\pi^2}$

$$\Longrightarrow \bar{u}(\mathrm{p}, \mathrm{s}) = \frac{k}{(\mathrm{k}\ p^2 - \mathrm{s})}\bar{u}_x(0, \mathrm{s}) - \frac{\pi}{(\mathrm{k}\ p^2 - \mathrm{s})(\mathrm{p}^2+\pi^2)}$$

$$= \frac{1}{(p^2 - \frac{\mathrm{s}}{\mathrm{k}})}\bar{u}_x(0, \mathrm{s}) - \frac{\pi}{\mathrm{k}\left(p^2 - \frac{\mathrm{s}}{\mathrm{k}}\right)(\mathrm{p}^2+\pi^2)}$$

$$= \frac{1}{(p^2 - \frac{\mathrm{s}}{\mathrm{k}})}\bar{u}_x(0, \mathrm{s}) - \frac{\pi}{\mathrm{k}\left(\frac{\mathrm{s}}{\mathrm{k}} + \pi^2\right)}\left\{\frac{1}{\left(p^2 - \frac{\mathrm{s}}{\mathrm{k}}\right)} - \frac{1}{(\mathrm{p}^2+\pi^2)}\right\}$$

$$= \frac{1}{(p^2 - \frac{\mathrm{s}}{\mathrm{k}})}\bar{u}_x(0, \mathrm{s}) - \frac{\pi}{(\mathrm{s} + \mathrm{k}\ \pi^2)\left(\mathrm{p}^2 - \frac{\mathrm{s}}{\mathrm{k}}\right)} + \frac{\pi}{(\mathrm{s} + \mathrm{k}\ \pi^2)(\mathrm{p}^2+\pi^2)}$$

$$= \frac{1}{\left(p-\sqrt{\frac{s}{k}}\right)\left(p+\sqrt{\frac{s}{k}}\right)} \left\{\bar{u}_x(0,\mathrm{s}) - \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}\right\} + \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)(\mathrm{p}^2+\pi^2)}$$

$$= \left\{\frac{1}{p-\sqrt{\frac{s}{k}}} - \frac{1}{p+\sqrt{\frac{s}{k}}}\right\} \frac{k}{2s} \left\{\bar{u}_x(0,\mathrm{s}) - \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}\right\} + \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)(\mathrm{p}^2+\pi^2)}$$

Applying L_x^{-1} , we get

$$\Rightarrow \bar{u}(\mathrm{x},\mathrm{s}) = \left\{e^{\left(\sqrt{\frac{s}{k}}\right)x} - e^{\left(\sqrt{\frac{s}{k}}\right)x}\right\} \frac{k}{2s} \left\{\bar{u}_x(0,\mathrm{s}) - \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}\right\} + \frac{1}{(\mathrm{s}+\mathrm{k}\,\pi^2)} \sin\pi\mathrm{x} \qquad (7)$$

Taking limit as $x \to 1$

$$\Rightarrow \bar{u}(1,\mathrm{s}) = \left\{e^{\left(\sqrt{\frac{s}{k}}\right)} - e^{\left(\sqrt{\frac{s}{k}}\right)}\right\} \frac{k}{2s} \left\{\bar{u}_x(0,\mathrm{s}) - \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}\right\} + \frac{1}{(\mathrm{s}+\mathrm{k}\,\pi^2)} \sin\pi$$

But $u(1, t) = 0 \Rightarrow \bar{u}(1, s) = 0$

$$\Rightarrow 0 = \left\{e^{\left(\sqrt{\frac{s}{k}}\right)} - e^{\left(\sqrt{\frac{s}{k}}\right)}\right\} \frac{k}{2s} \left\{\bar{u}_x(0,\mathrm{s}) - \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}\right\}$$

$$\Rightarrow \bar{u}_x(0,\mathrm{s}) = \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}$$

$$(7) \Rightarrow \bar{u}(\mathrm{x},\mathrm{s}) = \left\{e^{\left(\sqrt{\frac{s}{k}}\right)x} - e^{\left(\sqrt{\frac{s}{k}}\right)x}\right\} \frac{k}{2s} \left\{\frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)} - \frac{\pi}{(\mathrm{s}+\mathrm{k}\,\pi^2)}\right\} + \frac{1}{(\mathrm{s}+\mathrm{k}\,\pi^2)} \sin\pi\mathrm{x}$$

$$\Rightarrow \bar{u}(\mathrm{x},\mathrm{s}) = \frac{1}{(\mathrm{s}+\mathrm{k}\,\pi^2)} \sin\pi\mathrm{x}$$

Applying L_t^{-1} , we get

$$\Rightarrow u(\mathrm{x},\mathrm{t}) = \mathrm{e}^{-\mathrm{k}\pi^2\mathrm{t}} \sin\pi\mathrm{x} \qquad (8)$$

The problem can be interpreted physically. The given equation is the heat equation, where u(x, t) gives the temperature at a point x at time t. Consider a section bounded by planes x=0 & x=1 & the boundary conditions u(0, t)=0=u(1, t) give the temperature zero at the planes. The condition u(x, 0) = sinπx indicates the initial temperature in $0 < x < 1$. The u in (8) represents the temperature at time $t > 0$.

3.3 Double Laplace Transform & One Dimensional Wave Equation

Example: Semi-infinite String

Find the displacement w(x,t) of an elastic string subject to the following conditions:

i) The string is initially at rest on the x-axis from x=0 to ∞.

ii) For time t>0 the left end of the string is moved in a given fashion, namely,

$$w(0,t) = f(t) = \begin{cases} \sin t & \text{if } 0 \le t \le 2\pi \\ 0 & otherwise \end{cases}$$

iii) Furthermore $\lim_{x\to\infty} w(x,t) = 0$ for t≥0.

Of course, there is no infinite string, but our model describes a long string or rope (of negligible weight) with its right end fixed far out on the x-axis.

Solution:

We have to solve the wave equation $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$

For positive x & t, subject to the boundary conditions

$w(0,t) = f(t)$, $\lim_{x\to\infty} w(x,t) = 0$ (t≥0)

with f as given above, & the initial conditions $w(x,0) = 0$, $\frac{\partial w}{\partial t}\Big|_{t=0} = 0$

we use the Double Laplace Transform on wave equation

$$L_t L_x \{\frac{\partial^2 w}{\partial t^2}\} = c^2 L_t L_x \{\frac{\partial^2 w}{\partial x^2}\}$$

$$s^2 \overline{w}(p, s) - s\,\overline{w}(p, 0) - \overline{w}_t(p, 0) = c^2 \{ s^2\,\overline{w}(p,s) - p\,\overline{w}(0,s) - \overline{w}_x(0,s) \}$$

Given: $w(x, 0) = 0 \Longrightarrow \overline{w}(p, 0) = 0$ & $\frac{\partial w}{\partial t}\Big|_{t=0} = 0 \Longrightarrow \overline{w}_t(p, 0) = 0$

$$s^2\,\overline{w}(p, s) = c^2 \{ p^2\,\overline{w}(p,s) - p\,\overline{w}(0,s) - \overline{w}_x(0,s) \}$$

$$(c^2 p^2 - s^2)\,\overline{w}(p, s) = c^2 p\,\overline{w}(0,s) + c^2 \overline{w}_x(0,s)$$

$$\overline{w}(p, s) = \frac{c^2 p}{(c^2 p^2 - s^2)}\,\overline{w}(0,s) + \frac{c^2}{(c^2 p^2 - s^2)}\,\overline{w}_x(0,s) \qquad (9)$$

Now, $\overline{w}(0,s) = L_t\{w(0,t)\} = L_t\{f(t)\} = \bar{f}(s)$ &

$$\overline{\mathrm{w}}_{\mathrm{x}}(0,\mathrm{s}) = \lim_{p\to 0}\overline{\mathrm{w}}_{\mathrm{x}}(\mathrm{p},\mathrm{s}) = \lim_{p\to 0}\; L_t\, L_x\;\{\mathrm{w}_{\mathrm{x}}(\mathrm{x},\mathrm{t})\} =$$

$$\lim_{p\to 0}\int_0^\infty e^{-st}\left\{\int_0^\infty e^{-px}\;\mathrm{w}_{\mathrm{x}}(\mathrm{x},\mathrm{t})\mathrm{dx}\right\}dt$$

$$= \int_0^\infty e^{-st}\left\{\int_0^\infty \mathrm{w}_{\mathrm{x}}(\mathrm{x},\mathrm{t})\mathrm{dx}\right\}dt = \int_0^\infty e^{-st}\,\{\lim_{x\to\infty} w(x,t) - w(0,t)\}dt$$

$$= -\int_0^\infty e^{-st}\,w(0,t)dt = -\bar{f}(s)$$

From (9), $\bar{w}\,(\mathrm{p},\mathrm{s}) = \frac{\mathrm{c}^2\mathrm{p}}{(\mathrm{c}^2\,\mathrm{p}^2-\mathrm{s}^2)}\;\bar{f}(s) - \frac{\mathrm{c}^2}{(\mathrm{c}^2\,\mathrm{p}^2-\mathrm{s}^2)}\;\bar{f}(s)$

$$\bar{w}\,(\mathrm{p},\mathrm{s}) = \left\{\frac{1}{2}\left[\frac{1}{p-\frac{s}{c}} + \frac{1}{p+\frac{s}{c}}\right] - \frac{2c}{s}\left[\frac{1}{p-\frac{s}{c}} - \frac{1}{p+\frac{s}{c}}\right]\right\}\bar{f}(s)$$

By applying $L_x^{\;-1}$, we get

$$\bar{w}\,(\mathrm{x},\mathrm{s}) = \left\{\frac{1}{2}\left[e^{\frac{s}{c}x} + e^{-\frac{s}{c}x}\right] - \frac{2c}{s}\left[e^{\frac{s}{c}x} - e^{-\frac{s}{c}x}\right]\right\}\bar{f}(s)$$

$$\bar{w}\,(\mathrm{x},\mathrm{s}) = \left\{A(s)e^{\frac{s}{c}x} + B(s)\,e^{-\frac{s}{c}x}\right\}\bar{f}(s) \tag{10}$$

where $A(s) = \frac{1}{2}\left(1-\frac{c}{s}\right)$ & $B(s) = \frac{1}{2}\left(1+\frac{c}{s}\right)$

Again since $\lim_{x\to\infty} w(x,t) = 0$ for $t \geq 0$

Therefore,

since $\lim_{x\to\infty}\bar{w}(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st}\,w(x,t)dt = \int_0^\infty e^{-st}\lim_{x\to\infty} w(x,t)dt = 0$

This implies A(s) = 0 in (10) because c>0, so that for every fixed positive s the function $e^{\frac{s}{c}x}$ increases as x increases.

Therefore, (10) ⇒ $\bar{w}\,(\mathrm{x},\mathrm{s}) = \left\{B(s)\,e^{-\frac{s}{c}x}\right\}\bar{f}(s)$

Since A(s) = 0 ⟹ c = s Therefore $B(s) = \frac{1}{2}\left(1+\frac{s}{s}\right) = 1$

$$\Longrightarrow \bar{w}\,(\mathrm{x},\mathrm{s}) = \bar{f}(s)\;e^{-\frac{s}{c}x}$$

Taking L_t^{-1}, From second shifting theorem, we obtain

$w(x,t) = f(t-\frac{x}{c})\,u(t-\frac{x}{c})$

w(x, t) = sin(t - $\frac{x}{c}$) if $\frac{x}{c} <$ t $< \frac{x}{c} + 2\pi$ or ct$> x > (t - 2\pi)c$

and zero otherwise.

This is a single sine wave travelling to the right with speed c.

Note that a point x remains at rest until t $= \frac{x}{c}$, the time needed to reach that x if one starts at t = 0 (starts of the motion of the left end) & travels with speed c.

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