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Dynamic Response of Beam on a New Non-Uniform Dynamic Foundation Subjected to a Moving Vehicle Using Finite Element Method

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***Abstract*-- The purpose of paper analyzes influence of foundation mass on dynamic response of beam on non-uniform foundation subjected to a moving vehicle. This foundation model includes non-uniform linear elastic springs, shear layer, viscous damping and special consideration of the influence of a characteristic parameter of foundation mass. The moving vehicle is assumed to be consisted of two nodal masses that are connected by means of a spring-damper components. The equation of motion for the beam-vehicle-foundation interaction element is derived by means of dynamic balance principle. After by assembling the stiffness, damping and mass matrices, and vectors of nodal loads of all elements based on finite element method, the governing equations of motion for the integrated system are obtained and solved by step-by-step integration method procedure. The accuracy of the algorithm is verified by comparing the numerical results with the other numerical results in the literature. Also, the effects of characteristic parameter of foundation mass on dynamic analysis of beam-vehicles interaction are investigated detail. The results show that the influence of foundation mass has effects significantly on dynamic response of beam-vehicle interaction and more increasing dynamic response than others without influence of foundation mass.**

Keywords-- Foundation mass; moving vehicle; non-uniform foundation; dynamic analysis of beam-vehicle interaction

I. INTRODUCTION

One of the most fundamental foundations suggested quite early is Winkler model in 1867. It has been commonly used in engineering application and attracted attention of many researchers in during many last decades with the uniform or non-uniform foundation stiffness considered as linear or nonlinear elastic springs [1-6].

But, one of the most important deficiencies of the Winkler model is appearance a displacement discontinuity between the loaded and the unloaded part of the foundation surface. Hence, several other foundation models had proposed by introducing some kind of interaction between the in-dependent springs by visualising various types of interconnections to overcome the deficiency of Winkler model such as: Filonenko [7]; Hetenyi, [8]; Pasternak [9]; Reissener [10]; Kerr [11]; Vlasov [12].

It can be seen that the foundation always has foundation mass in reality, so that the foundation mass have to effect on dynamic response of structure-foundation interaction in during vibration of its. But, one of the most important deficiencies of the above foundation models overlooks the influence of the foundation mass. Hence, a new foundation model called dynamic foundation model including elastic spring, shear layer, viscous damping and special consideration of the influence of a characteristic parameter of foundation mass had proposed by Pham [13]. The dynamic foundation model applied to analyze response of beam and plate structures subjected to moving load [14, 15] and the results show that the influence of foundation mass has effects significantly on dynamic response of structures.

To continuously attention to effects of foundation mass on dynamic responses of structures, this study analyzes effects of the foundation mass on dynamic response of beam subjected to a moving vehicle. This foundation model includes the non-uniform linear elastic springs, the uniform linear shear layer, viscous damping and special consideration of the influence of a characteristic parameter of foundation mass. The moving vehicle is assumed to be consisted of two nodal masses that are connected by means of a spring-damper components. By means of dynamic balance principle and finite element method, the equation of motion for the beam-vehicle-foundation interaction element is derived. The governing equations of motion for the integrated system are obtained by assembling the stiffness, damping and mass matrices, and vectors of nodal loads of all elements, and solved by step-by-step integration method procedure such as Newmark's method. The accuracy of the algorithm is verified by comparing the numerical results with the other numerical results in the literature. Also, the effects of characteristic parameter of foundation mass on dynamic analysis of beam-vehicle interaction are discussed.

II. FORMULATION

A. Model of beam-vehicle-foundation interaction

Consider an Euler-Bernoulli beam resting on the dynamic foundation subjected to moving vehicles is shown in Fig.1.

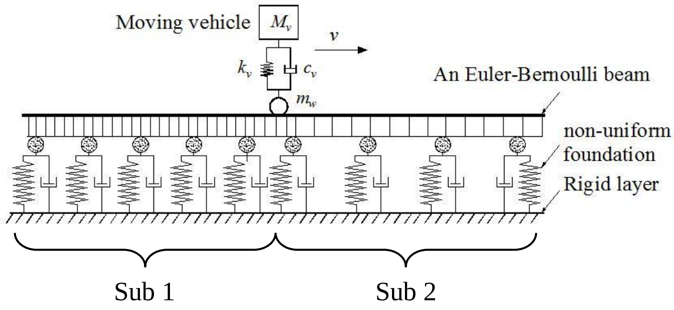

Fig1. The beam subjected to moving vehicle on non-uniform foundation.

The non-uniform foundation based on the dynamic foundation model [13], which fully describes dynamic characteristic parameters for behavior of foundation including the elastic stiffness idealized based on the linear elastic springs modulus k_W, the shear foundation modulus k_S, viscous damping c and the foundation mass ρ_F are respectively replaced by lumped mass m at the top of the elastic spring connected between elastic layer and shear layer. The pressure-deflection relationship at the time t due to a pressure $q(x,y,t)$ is determined based on dynamic balance principle, can be expressed mathematically as follows

$$q(x,y,t)=k_W w(x,y,t)+c\frac{\partial w(x,y,t)}{\partial t}+ m\frac{\partial^2 w(x,y,t)}{\partial t^2}-k_s\nabla^2 w(x,y,t) \tag{1}$$

where, the lumped mass m is given by

$$m=\beta\rho_f \tag{2}$$

in which β is an experimental parameter characterized the influence of foundation mass.

B. Formulation of element matrices

The beam modeled as uniform Euler-Bernoulli beam is assumed that the beam material is isotropic; the vibration amplitudes of beam are sufficiently small and the bond between the beam and the foundation is perfect. Each beam element has two nodes, each node having two degree of freedom including vertical displacement and rotation displacement.

Based on the strain energy of the beam element, stiffness matrix of the beam element resting on non-uniform foundation including the effects of both bending deformation of the beam and non-uniform foundation is given by

$$[\mathbf{K}]_e=[\mathbf{K}]_e^B+[\mathbf{K}]_e^W+[\mathbf{K}]_e^S \tag{3}$$

where $[\mathbf{K}]_e^B$ is the normal bending stiffness matrix; $[\mathbf{K}]_e^W$ and $[\mathbf{K}]_e^S$ are the non-uniform elastic stiffness matrix and the shear stiffness matrix, respectively, are given by

$$\begin{aligned}[\mathbf{K}]_e^W&=\int[\mathbf{N}_w]^T k_W[\mathbf{N}_w]dx\\ [\mathbf{K}]_e^S&=\int[\mathbf{N}_s]^T k_S[\mathbf{N}_s]dx\end{aligned} \tag{4}$$

The mass matrix of the beam element including the effects of the foundation mass of both the beam and foundation based on the kinetic energy of the beam element, can be expressed as

$$[\mathbf{M}]_e=[\mathbf{M}]_e^b+[\mathbf{M}]_e^F \tag{5}$$

where $[\mathbf{M}]_e^b$ is the mass matrix of the beam element and $[\mathbf{M}]_e^F$ is the mass matrix for the influence of the foundation mass, is also written as

$$[\mathbf{M}]_e^F=\int[\mathbf{N}_w]^T m[\mathbf{N}_w]dx \tag{6}$$

The viscous damping property of the foundation is considered to be the dashpots system and based on the dissipated energy of these disputes the damping matrix can be expressed as

$$[\mathbf{C}]_e^F=\int[\mathbf{N}_w]^T c[\mathbf{N}_w]dx \tag{7}$$

where $[\mathbf{N}_s]$ and $[\mathbf{N}_w]$ are the matrix of interpolation functions for displacements and rotation, respectively, studied in many research related to finite element method.

C. The governing equation of motion

The moving vehicle model is regarded as a two-node with one is associated with each of two concentrated masses. The stiffness and damping coefficients of the oscillator are denoted by k_v and c_v, respectively. The mass of the vehicle and the mass of the wheel is denoted by M_v and mw, respectively. In addition, z_v and z_w denote the vertical displacements of two nodes measured from the static equilibrium position. At any time t, the position of the moving vehicle is $x_m=vt$ and the left end of the beam element in global coordinate (node i^{th}) is to be $x_i=Int[x_m/l]l$. Then, one can find the element number $i^{th}=\mathrm{Int}[x_m/l]+1$, nodes i^{th} and $(i+1)^{th}$, which the moving vehicle is applied to at any time t. Therefore, ξ can be rewritten in terms of the global instead of the local

$$\xi(t)=x_m-i^{th}l \tag{8}$$

By assuming the no-jump condition for the moving vehicle, the contact force can be related to displacement of the contact force and its derivatives. Equations of motion of the moving vehicle can be written as follows

$$\begin{bmatrix}M_v & 0\\ 0 & m_w\end{bmatrix}\begin{Bmatrix}\ddot{z}_v\\ \ddot{z}_w\end{Bmatrix}+\begin{bmatrix}c_v & -c_v\\ -c_v & c_v\end{bmatrix}\begin{Bmatrix}\dot{z}_v\\ \dot{z}_w\end{Bmatrix}+ \begin{bmatrix}k_v & -k_v\\ -k_v & k_v\end{bmatrix}\begin{Bmatrix}z_v\\ z_w\end{Bmatrix}=\begin{Bmatrix}0\\ f_c-(M_v+m_w)g\end{Bmatrix} \tag{9}$$

where f_c is the contact force.

Assuming that all information of the system at time t is known and Δt is a small time increment, the first row of Eq. (9) can be expanded in an incremental form at time $t+\Delta t$ [16]

$$M_v\ddot{z}_{v,t+\Delta t}+c_v\dot{z}_{v,t+\Delta t}+k_v z_{v,t+\Delta t}=q_{vc,t+\Delta t} \tag{10}$$

with

$$q_{vc,t+\Delta t} = -c_v \dot{z}_{w,t+\Delta t} - k_v z_{w,t+\Delta t} \quad (11)$$

Based on Newmark's method, average acceleration method ($\gamma = 0.5$ and $\beta = 0.25$), the displacement z_w and its derivatives at time $t+\Delta t$ can be written as

$$\begin{aligned} \ddot{z}_{v,t+\Delta t} &= \frac{b_0}{\psi_v}\left(-q_{vc,t+\Delta t} + q_{v,t}\right) - b_1 \dot{z}_{v,t} - b_2 \ddot{z}_{v,t} \\ \dot{z}_{v,t+\Delta t} &= \frac{b_3}{\psi_v}\left(-q_{vc,t+\Delta t} + q_{v,t}\right) - b_4 \dot{z}_{v,t} - b_5 \ddot{z}_{v,t} \\ z_{v,t+\Delta t} &= z_{v,t} + \frac{1}{\psi_v}\left(-q_{vc,t+\Delta t} + q_{v,t}\right) \end{aligned} \quad (12)$$

with

$$\begin{aligned} \psi_v &= b_0 M_v + b_3 c_v + k_v \\ q_{v,t} &= M_v\left(b_1 \dot{z}_{v,t} + b_2 \ddot{z}_{v,t}\right) + c_v\left(b_4 \dot{z}_{v,t} + b_5 \ddot{z}_{v,t}\right) - k_v z_{v,t} \end{aligned} \quad (13)$$

and coefficients bi given by

$$\begin{aligned} & b_0 = \frac{1}{\beta \Delta t^2},\ b_1 = \frac{1}{\beta \Delta t},\ b_2 = \frac{1}{2\beta} - 1,\ b_3 = \frac{\gamma}{\beta \Delta t}, \\ & b_4 = \frac{\gamma}{\beta} - 1,\ b_5 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right) \end{aligned} \quad (14)$$

Substituting Eq. (12) into incremental form of the second row of Eq. (9), the contact force fc in time $t+\Delta t$ is determined by

$$f_{c,t+\Delta t} = m_w \ddot{z}_{w,t+\Delta t} + c_c \dot{z}_{w,t+\Delta t} + k_c z_{w,t+\Delta t} + p_{c,t+\Delta t} + q_{c,t} \quad (15)$$

in which

$$\begin{aligned} & c_c = c_v\left(1 + \frac{\psi_{vw}}{\psi_v}\right),\ k_c = k_v\left(1 + \frac{\psi_{vw}}{\psi_v}\right) \\ & p_{c,t+\Delta t} = \left(M_v + m_w\right) g,\ p_{c,t} = \frac{\psi_{vw}}{\psi_v} q_{v,t} - q_{w,t} \end{aligned} \quad (16)$$

with

$$\psi_{vw} = -b_3 c_v - k_v,\ q_{w,t} = -c_v\left(b_4 \dot{z}_{v,t} + b_5 \ddot{z}_{v,t}\right) + k_v z_{v,t} \quad (17)$$

It assumes that there is no loss of contact between the tire and the upper surface of the beam, the displacement and acceleration of the wheel (z_w and $\ddot{z}_w$, respectively) equals to the deflection and acceleration at the contact position of vehicle and beam. Therefore, the differential equation of motion of the beam element resting on the non-uniform foundation subjected to the moving vehicle at time t can be expressed as

$$[\mathbf{M}_e]\{\ddot{\mathbf{u}}_e\} + [\mathbf{C}_e]\{\dot{\mathbf{u}}_e\} + [\mathbf{K}_e]\{\mathbf{u}_e\} = \delta(x_i - vt)\left[\mathbf{N}_{w,\xi}\right] f_c \quad (18)$$

where $\left[\mathbf{N}_{w,\xi}\right]$ is the value of interpolation function depended on the coordinate ξ corresponding with the position of wheel on the beam element i^{th} at the time t; $\delta(x_i - vt)$ is the Dirac delta function and i denotes contact element between the beam and contact force.

By assembling the stiffness, damping and mass matrices, and vectors of nodal loads of all elements corresponding degrees of freedom in the global coordinate, the governing equation of motion of the system beam-vehicles-foundation interaction in each time step is defined as follows

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\} + [\mathbf{C}]\{\dot{\mathbf{U}}\} + [\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\} \quad (19)$$

where $[\mathbf{M}]$, $[\mathbf{C}]$, and $[\mathbf{K}]$ are the overall mass, damping and stiffness matrices of the system, respectively; $\{\mathbf{U}\}$ and $\{\mathbf{F}\}$ is the nodal displacement vector and the external force vector of the system, respectively. It is used for studying the dynamic response of the beam-vehicle-foundation interaction and solved by means of the direct step-by-step integration method based on Newmark's algorithm.

III. NUMERICAL RESULTS

A. Verified examples

The first example analyzes free vibration of a simple support beam resting on a uniform linear foundation with-out effect of foundation mass. The dimensionless parameters K_1 and K_2 representing the stiffness of linear elastic springs and shear layer of the foundation and the dimensionless natural frequency λ are defined as follows [17]

$$K_1 = \frac{k_W L^4}{EI},\ K_2 = \frac{k_S L^2}{\pi^2 EI},\ \lambda = \omega L^2 \sqrt{\frac{\rho A}{EI}} \quad (20)$$

in which ω is natural circular frequency of the beam. The convergences of the lowest natural frequencies are compared with the results of Matsunaga [17], shown in Table 1.

Table 1. The dimensionless natural frequencies λ of the beam

K_1	K_2	**Present**	**Ref. [17]**
0		13.9577	13.9577
10		14.3115	14.3115
10^2		17.1703	17.1703
10^3	1	34.5661	34.5661
10^4		100.9694	100.9694
10^5		316.5356	316.5356

Table 2. The maximum vertical displacement at the midpoint of the beam

k_1 **(kN/m²)**	k_2 **(kN/m²)**	**Present (mm)**	**SAP2000 (mm)**
125	250	3.8586	3.8593
250	500	2.3666	2.3674
500	1000	1.3830	1.3834
750	1500	1.0023	1.0026
1000	2000	0.7980	0.7982

Continuously, the maximum vertical displacement of a simply supported beam resting on non-uniform linear elastic foundation subjected to a vertical static point load at the midpoint of the beam is investigated. The beam has length $L = 20\,m$, Young's modulus $E = 210\,Gpa$, moment of inertia $I = 0.667\ 10^{-4}\,m^4$, mass per unit volume $\rho = 7800\,kg/m^3$ and section area $A = 0.2\,m^2$. The point force acting on the beam has a value of $P = 10\,kN$ and the non-uniform foundation composed of two sub-domains of equal length, the first (on the left side) with k_1 and the second (on the right side) with k_2, see in Fig. 1. The maximum of vertical displacement of the midpoint of the beam are compared with result obtained by SAP2000 software, shown in Table 2.

In the next example is carried out to verify the present algorithm for a simply supported beam subjected to a moving vehicle. The following data are adopted the beam and the moving vehicle similar to Neves et al., 2012 [18]. The time history of vertical displacement of midpoint of the beam and body of the moving vehicle are plotted in Fig. 2.

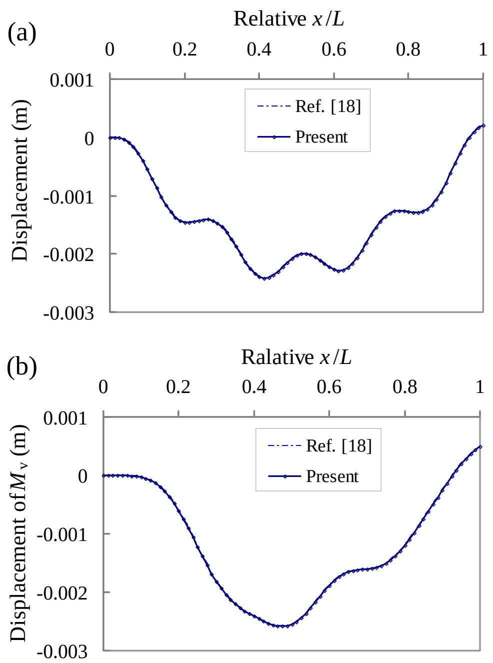

Fig 2. Time history of displacement: (a) Vertical displacement at the midpoint of beam; (b) Vertical displacement of body.

Through the above examples, the numerical results from the program based on the suggested formulation show quite good agreement with numerical results in the literature. Therefore, the program which will analyze the influence of many parameters on dynamic response of the beam-vehicle-foundation interaction is reliable.

B. Numerical investigation

In this section, the physical and geometric properties of a simply supported beam, vehicle and dynamic foundation analyzed the effects of foundation mass on dynamic response of beam-vehicle-foundation interaction are listed in Table 3. The characteristic parameter of elastic stiffness of non-uniform foundation κ is defined by ratio of elastic foundation stiffness of sub two (on the right) with elastic foundation stiffness of sub one (on the left), and the characteristic parameter of length of non-uniform foundation ζ is also defined by ratio of length of sub one (on the left) with length of sub two (on the right).

Table 3. Properties of railway track, vehicle and dynamic foundation

Item	Notation	Unit	Value
Beam			
Length	L	m	20
Young's modulus	E	Gpa	24
Foundation mass	ρ	kg/m^3	2500
Cross sectional area	A	m^2	0.3
Second moment of area	I	m^4	2.25x10^{-3}
Moving Vehicle			
Mass of car body	M_v	kg	5x10^3
Spring stiffness	k_v	N/m	1.5x10^6
Dashpot coefficient	c_v	Ns/m	1.5 x10^4
Mass of axle	m_w	kg	5x10^2
Non-uniform foundation			
Linear stiffness	k_W	N/m^2	1.5x10^6
Shear parameter	k_S	N	5x10^4
Viscous damping	c	Ns/m^2	1.5x10^3
Density of foundation	ρ_f	kg/m^3	1800

In the first investigation, the dynamic responses of the beam on the foundation subjected to a moving vehicle with various values of the characteristic parameter of non-uniform foundation are studied. The effects of characteristic parameters of elastic stiffness of non-uniform foundation on time history of vertical displacement of the midpoint of the beam have been plotted in Fig. 3 and 4. The influence of foundation mass on vertical displacements of the midpoint of the beam is plotted in Fig. 5. It can be seen that the characteristic parameter of the non-uniform foundation and foundation mass of the foundation effect significantly on dynamic behavior of the beam. It increases the time history of vertical displacements of the midpoint of the beam with decrease values of the parameters κ and ζ, shown in Fig 3 and Fig. 4. But, with an increase of values of the parameters of foundation mass also increase the time history of the vertical displacements of the midpoint of the beam, plotted in Fig. 5.

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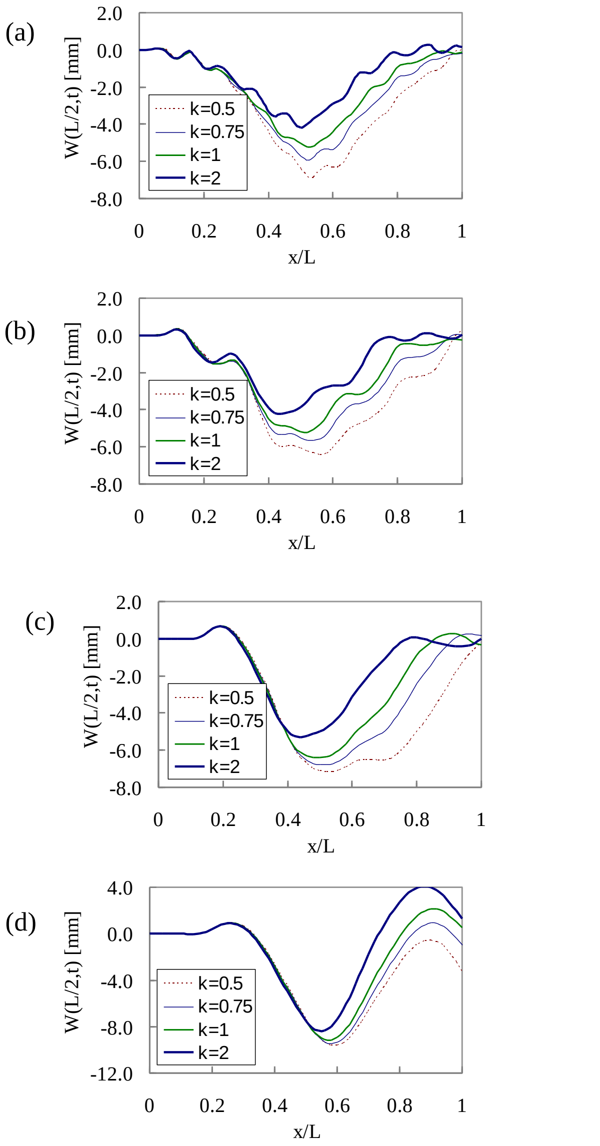
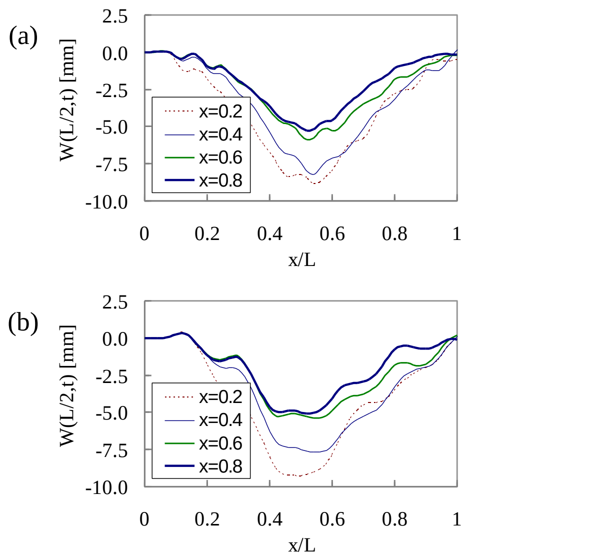


Fig 3. Time history of displacement with β=0.5 and ζ=0.5: (a) v=10 m/s, (b) v=25 m/s, (c) v=50 m/s, (d) v=75 m/s



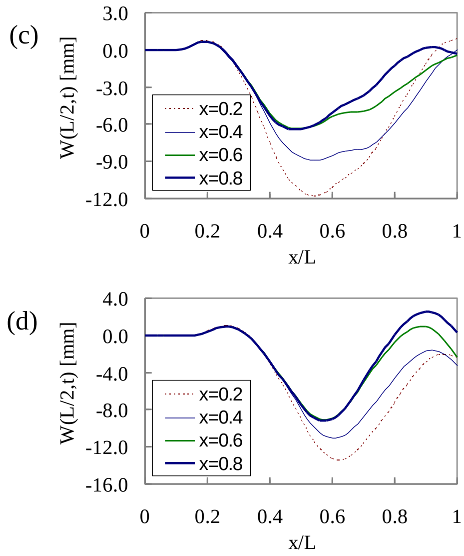


Fig 4. Time history of displacement with κ=0.5 and β=0.5: v=10 m/s, (b) v=25 m/s, (c) v=50 m/s, (d) v=75 m/s

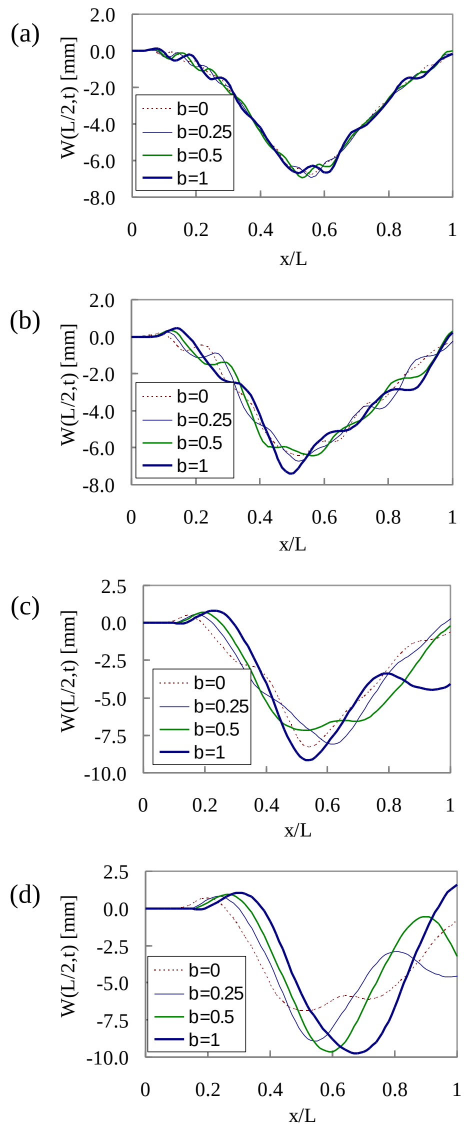


Fig 5. Time history of displacement with κ=0.5 and ζ=0.5: (a) v=10 m/s, (b) v=25 m/s, (c) v=50 m/s, (d) v=75 m/s

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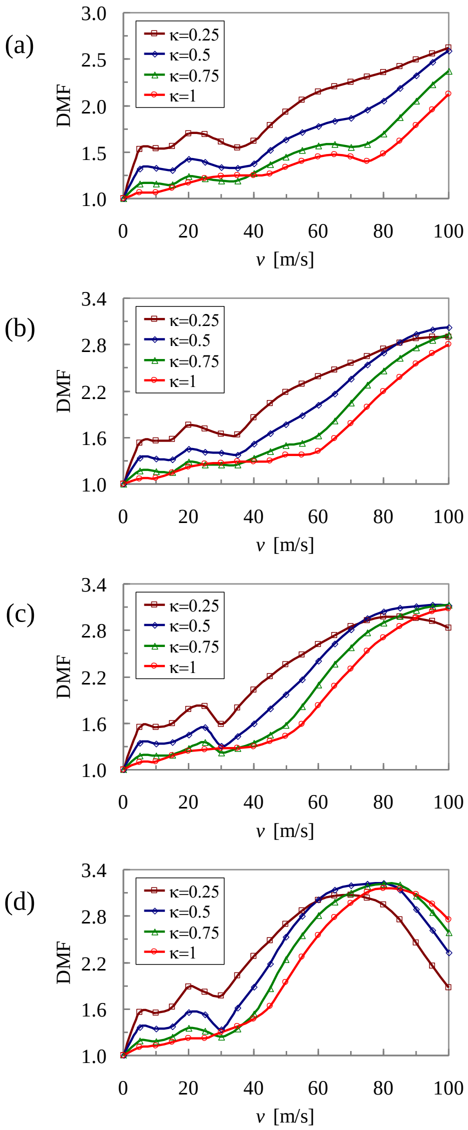
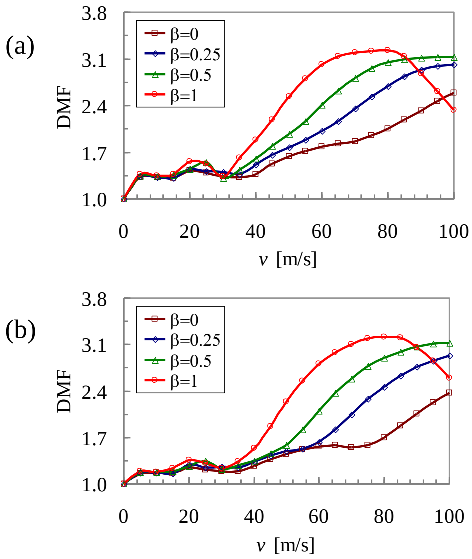


Fig 6. The effects of the parameter κ on the DMFs of the beam: (a) β=0, (b) β=0.25, (c) β=0.5, (d) β=1



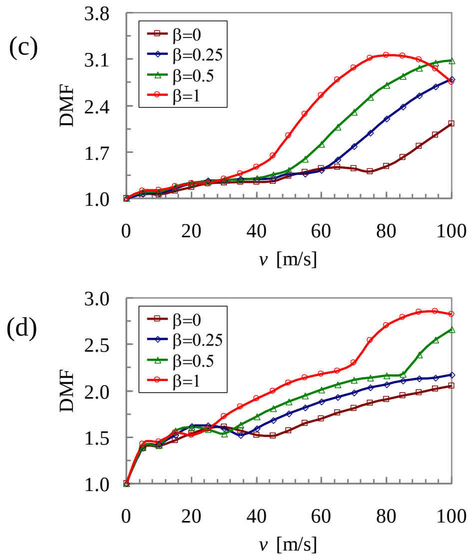


Fig 7. The effects of the parameter β on the DMFs of the beam: κ=0.5, (b) κ=0.75, (c) κ=1, (d) κ=2

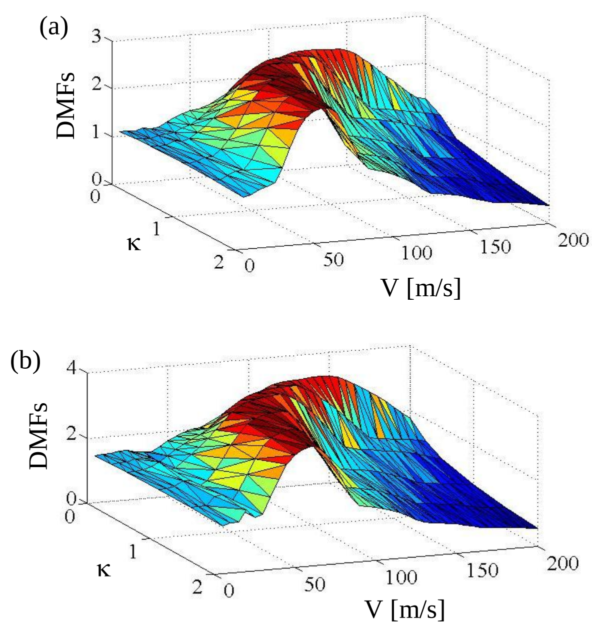
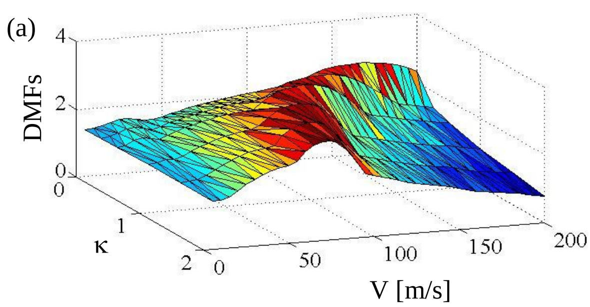


Fig 8. The effects of the parameter β and κ on the DMFs of vertical displacement of the beam: (a) ζ=0.3, (b) ζ=0.6



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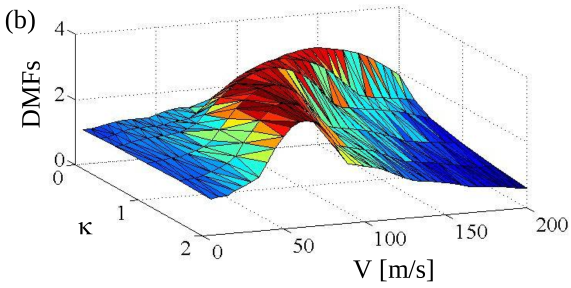

Fig 9. The effects of the parameter β and ζ on the DMFs of vertical displacement of the beam: (a) κ=0.5, (b) κ=2

To show more clear the influence of the characteristic parameters of the non-uniform foundation on dynamic analysis of the interaction between the beam and foundation subjected to a moving vehicle, the effects of the above parameters on dynamic magnification factor (DMF) are investigated for various values of the velocity of the moving vehicle, shown in Fig. 6 and Fig. 7. At the same time, the effects of the between parameters κ or ζ and the parameters of foundation mass β for various values of the velocity of the moving vehicle on DMFs of the beam are also studied, shown in Fig. 8 and Fig. 9.

It can be shown that the parameters of the non-uniform and the foundation mass affect significantly on the dynamic response of the beam, shown from Fig 6 to Fig. 9. In the range of low velocities, it increases clearly the DMFs of the beam with decrease of values of the characteristic parameters of elastic stiffness of the foundation. At the same time, in the range of high velocity, the effects of the foundation mass on the DMFs of vertical dis-placement of the beam are quite clear, and the comparisons show that the foundation mass is an increase the DMFs of the beam than the foundation model without the influence of foundation mass ($\beta = 0$).

IV. CONCLUSIONS

In this paper, the dynamic analysis of the beam on the non-uniform foundation subjected to a moving vehicle is investigated by means of finite element method. The non-uniform foundation includes non-uniform linear elastic spring, shear layer, viscous damping and special consideration influence of foundation mass of it. The beam, vehicle and non-uniform foundation are regarded as an integrated system and the governing equation of motion of the system is derived based on dynamic balance principle and solved by step-by-step integration method procedure. The accuracy of the numerical results is verified by comparing its numerical solutions with those of other available numerical results. The results show that the characteristic parameters of non-uniform and foundation mass effect significantly on dynamic response of the beam. A comparison shows that the foundation mass in the dynamic foundation model is more increasing dynamic behavior of the beam than others without the influence of foundation mass. The presented results can be employed to perform the parametric studies about various dynamic and structural properties of the structural-vehicle-foundation interaction model such as track-train-foundation, road-vehicle-foundation and it also is useful for problems of practical design.

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