Dynamical Behaviors of Discrete-time Prey-Predator System

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***Abstract*—In the present study, the dynamical behaviors of discrete-time prey-predator system. Global stability of the model at the fixed points has been discussed. The specific conditions for existence of flip bifurcation and Hopf bifurcation have been derived by using center manifold theorem and bifurcation theory. To analyse our results, numerical simulations have been carried out.**

***Keywords*—Prey-predator system, Center manifold theorem, Flip bifurcation, Hopf bifurcation, Chaos.**

I. INTRODUCTION

The Prey-Predator model is a topic of great interest for many mathematicians and biologists which starts with the pioneer work of Lotka [1] and Volterra [2]. The dynamic relationship between predators and prey living in the same environment will continue to be one of the important themes in mathematical ecology [3-4]. Many authors [5-11] have suggested that the discrete time models are more appropriate than the continuous ones and provide efficient results when the populations have non-overlapping generations. However, there are few articles [12-18] discussing the dynamical behaviors of discrete-time predator–prey models by involving bifurcations and chaos phenomena.

In the present study, we investigated the dynamical behaviors of discrete-time prey-predator model by involving bifurcations and chaos phenomena. This paper is organized as follows: In Section 2, we obtained the fixed points of the discrete-time model and discussed the stability criterion of the discrete-time model at fixed points. In Section 3, the specific conditions of existence of flip bifurcation and Hopf bifurcation have been derived. In Section 4, to analyse our results, numerical simulations have been carried out and further discussion on the period doubling bifurcation and chaotic behavior has been carried out.

The discrete-time prey-predator model [19] is

$$\begin{cases} \frac{dx}{dt} = x(a - cx) - lxy, \\ \frac{dy}{dt} = y(-b - dy) + mxy, \end{cases} \qquad (1.1)$$

where x and y represents the densities of prey and predator respectively; a, l, b, m denotes the intrinsic growth rate of prey, capture rate, death rate of predator and the conversion rate respectively; and c, d denotes the intra-specific competition coefficients of prey and predator respectively.

Applying forward Euler's scheme to the system of equations (1.1), we obtain the system as

$$\begin{cases} x \to x + \delta[x(a - cx) - lxy], \\ y \to y + \delta[y(-b - dy) + mxy]. \end{cases} \qquad (1.2)$$

II. STABILITY OF THE FIXED POINTS

The fixed points of the system (1.2) are $O(0,0), A\left(\frac{a}{c}, 0\right), B(x^*, y^*)$,

where $x^* = \frac{ad+bl}{lm+cd}, y^* = \frac{am-bc}{lm+cd}$.

The jacobian matrix of (1.2) at the fixed point (x, y) is given by

$$J = \begin{bmatrix} 1 + \delta(a - 2cx - ly) & -l\delta x \\ \delta my & 1 + \delta(-b - 2dy + mx) \end{bmatrix}.$$

The characteristic equation of the jacobian matrix can be written as

$$\lambda^2 + p(x, y)\lambda + q(x, y) = 0, \qquad (2.1)$$

where

$$p(x, y) = -trJ = -2 - \delta a + \delta b + \delta x(2c - m) + \delta y(l + 2d),$$

$$q(x, y) = detJ = [1 + \delta(a - 2cx - ly)][1 + \delta(-b - 2dy + mx)] + \delta^2 lmxy$$

Lemma 2.1: Let $F(\lambda) = \lambda^2 + B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are roots of $F(\lambda) = 0$. Then

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;

(ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) if and only if* $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$;

(iv) *$\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;*

(v) *λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.*

Let λ_1 and λ_2 be the roots of eq. (2.1), which are known as eigen values of the fixed point (x, y). Then (x, y) is called a sink or locally asymptotically stable if $|\lambda_1| < 1$and $|\lambda_2| < 1$. (x, y) is called a source or locally unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. (x, y) is non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$. (x, y) is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). (see [11])

Proposition 2.2. *The fixed point $O(0,0)$ is source if $\delta > \frac{2}{b}$, saddle if $0 < \delta < \frac{2}{b}$, and non-hyperbolic if $\delta = \frac{2}{b}$.*

It has been observed that when $\delta = \frac{2}{b}$, one of the eigen values of the critical point $O(0,0)$ is -1 and magnitude of other is not equal to 1. Thus the flip bifurcation occur when parameter changes in small neighborhood of $\delta = \frac{2}{b}$.

Proposition 2.3. *There exists different topological types of $A\left(\frac{a}{c}, 0\right)$ for possible parameters.*

(i) *$A\left(\frac{a}{c}, 0\right)$ is sink if $bc > am$ and $0 < \delta < min\left\{\frac{2}{a}, \frac{2c}{bc-am}\right\}$.*

(ii) *$A\left(\frac{a}{c}, 0\right)$ source if $bc > am$ and $\delta > max\left\{\frac{2}{a}, \frac{2c}{bc-am}\right\}$.*

(iii) *$A\left(\frac{a}{c}, 0\right)$ is non-hyperbolic if $\delta = \frac{2}{a}$ or $\delta = \frac{2c}{bc-am}$ and $bc > am$.*

(iv) *$A\left(\frac{a}{c}, 0\right)$ is saddle for all values of the parameters, except for that which lies in (i) to (iii).*

The term (iii) of proposition 2.3 implies that the parameters lie in the set

$$F_A = \left\{(a, b, c, m, \delta), \delta = \frac{2}{a}, \delta \neq \frac{2c}{bc-am} \text{ and } bc \neq am, a, b, c, m, \delta > 0\right\}.$$

If the term (iii) of proposition 2.3 holds, than one of the eigen values of the fixed point $A\left(\frac{a}{c}, 0\right)$ is -1 and magnitude of other is not equal to 1. The point $A\left(\frac{a}{c}, 0\right)$ undergoes flip bifurcation when the parameter changes in small neighbourhood of F_A.

Proposition 2.4. *When $am > bc$, there exists different topological types of $B(x^*, y^*)$,* where $x^* = \frac{ad+bl}{lm+cd}, y^* = \frac{am-bc}{lm+cd}$ *for all possible parameters.*

(i) *$B(x^*, y^*)$ is sink if either condition (i.1) or (i.2) holds:*

(i.1) $0 < \delta < \frac{N-\sqrt{M}}{(ad+bl)(am-bc)}, M \geq 0.$

(i.2) $0 < \delta < \frac{N}{(ad+bl)(am-bc)}$, $M < 0,$

where $N = acd + adm + bcl - bcd$ and $M = N^2 - 4(ad + bl)(am - bc)(lm + cd)$.

(ii) *$B(x^*, y^*)$ is source if either condition (ii.1) or (ii.2) holds:*

(ii.1) $\delta > \frac{N+\sqrt{M}}{(ad+bl)(am-bc)}$, $M \geq 0.$

(ii.2) $\delta > \frac{N+\sqrt{M}}{(ad+bl)(am-bc)}$, $M < 0.$

(iii) *$B(x^*, y^*)$ is non-hyperbolic if either condition (iii.1) or (iii.2) holds:*

(iii.1) $\delta = \frac{N\pm\sqrt{M}}{(ad+bl)(am-bc)}, M \geq 0.$

(iii.2) $\delta = \frac{N\pm\sqrt{M}}{(ad+bl)(am-bc)}, M < 0.$

(iv) *$B(x^*, y^*)$ is saddle for all values of the parameters, except for that which lies in (i) to (iii).*

From lemma (2.1), it has been observed that one of the eigen values of the fixed point $B(x^*, y^*)$ is -1 and magnitude of other is not equal to 1, if the term (iii.1) of proposition 2.4 holds. The term (iii.1) of proposition 2.4 may be written as follows:

$$F_{B1} = \left\{(a, b, c, d, l, m): \delta = \frac{N-\sqrt{M}}{(ad+bl)(am-bc)}, M \geq 0, am - bc > 0, a, b, c, d, l, m > 0\right\}, \quad F_{B2} = \left\{(a, b, c, d, l, m): \delta = \frac{N+\sqrt{M}}{(ad+bl)(am-bc)}, M \geq 0, am - bc > 0, a, b, c, d, l, m > 0\right\},$$

where $N = acd + adm + bcl - bcd$ and $M = N^2 - 4(ad + bl)(am - bc)(lm + cd)$.

From lemma (2.1), it has been observed that the eigen values of the fixed point $B(x^*, y^*)$ as a pair of conjugate complex numbers with modulus 1, if the term (iii.2) of proposition 2.4 holds. The term (iii.2) of proposition 2.4 may be described as follows:

$$H_B = \left\{(a,b,c,d,l,m): \delta = \frac{N}{(ad+bl)(am-bc)}, M < 0, am - bc > 0, a,b,c,d,l,m > 0\right\}.$$

III. BIFURCATION BEHAVIOR

In this section, we study the flip bifurcation and Hopf bifurcation at the fixed point $B(x^*, y^*)$.

III.1 FLIP BIFURCATION

Consider the system (1.2) with arbitrary parameter $(a_1, b_1, c_1, d_1, l_1, m_1, \delta_1) \in F_{B1}$, which is described as follows:

$$\begin{cases} x \to x + \delta_1[x(a_1 - c_1x) - l_1xy], \\ y \to y + \delta_1[y(-b_1 - d_1y) + m_1xy]. \end{cases} \quad (3.1)$$

Eq. (3.1) has fixed point $B(x^*, y^*)$, whose eigen values are $\lambda_1 = -1$, $\lambda_2 = 3 - \frac{\delta_1(a_1c_1d_1 + a_1d_1m_1 + b_1c_1l_1 - b_1c_1d_1)}{(l_1m_1 + c_1d_1)}$ with $|\lambda_2| \neq 1$ by proposition (2.4), where $x^* = \frac{a_1d_1 + b_1l_1}{l_1m_1 + c_1d_1}$, $y^* = \frac{a_1m_1 - b_1c_1}{l_1m_1 + c_1d_1}$ and

$$\delta_1 = \frac{N_1 - \sqrt{M_1}}{(a_1d_1 + b_1l_1)(a_1m_1 - b_1c_1)}.$$

where $N_1 = a_1c_1d_1 + a_1d_1m_1 + b_1c_1l_1 - b_1c_1d_1$ and $M_1 = {N_1}^2 - 4(a_1d_1 + b_1l_1)(a_1m_1 - b_1c_1)(l_1m_1 + c_1d_1)$.

Consider the perturbation of (3.1) as below:

$$\begin{cases} x \to x + (\delta_1 + \delta^*)[x(a_1 - c_1x) - l_1xy], \\ y \to y + (\delta_1 + \delta^*)[y(-b_1 - d_1y) + m_1xy], \end{cases} \quad (3.2)$$

where $|\delta^*| \ll 1$ is a limited perturbation parameter.

Let $u = x - x^*$ and $v = y - y^*$.

After transformation of the fixed point $B(x^*, y^*)$ of map (3.2) to the point (0, 0), we obtained

$$u \to a_{11}u + a_{12}v + a_{13}uv + a_{14}u^2 + b_{11}\delta^*u + b_{12}\delta^*v + b_{13}\delta^*uv + b_{14}\delta^*u^2 \quad \text{and} \quad v \to a_{21}u + a_{22}v + a_{23}uv + a_{24}v^2 + b_{21}\delta^*u + b_{22}\delta^*v + b_{23}\delta^*uv + b_{24}\delta^*v^2, \quad (3.3)$$

where

$a_{11} = 1 + \delta_1[a_1 - 2c_1x^* - l_1y^*]$, $a_{12} = -l_1\delta_1x^*$, $a_{13} = -l_1\delta_1$, $a_{14} = -c_1\delta_1$,

$b_{11} = a_1 - 2c_1x^* - l_1y^*$, $b_{12} = -l_1x^*$, $b_{13} = -l_1$, $b_{14} = -c_1$,

$a_{21} = m_1\delta_1y^*$, $a_{22} = 1 + \delta_1[-b_1 - 2d_1y^* + m_1x^*]$, $a_{23} = m_1\delta_1$, $a_{24} = -d_1\delta_1$,

$b_{21} = m_1y^*$, $b_{22} = -b_1 - 2d_1y^* + m_1x^*$, $b_{23} = m_1$, $b_{24} = -d_1$.

Consider the following translation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},$$

where $T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}$.

Taking T^{-1} on both sides of eq. (3.3), we get

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} f(u,v,\delta^*) \\ g(u,v,\delta^*) \end{pmatrix}, \quad (3.4)$$

where

$$f(u,v,\delta^*) = \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]uv}{a_{12}(\lambda_2+1)} + \frac{[a_{14}(\lambda_2 - a_{11})]u^2}{a_{12}(\lambda_2+1)} - \frac{a_{12}a_{24}v^2}{a_{12}(\lambda_2+1)} + \frac{[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]\delta^*u}{a_{12}(\lambda_2+1)} + \frac{[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}]\delta^*v}{a_{12}(\lambda_2+1)} + \frac{[b_{13}(\lambda_2 - a_{11}) - a_{12}b_{23}]\delta^*uv}{a_{12}(\lambda_2+1)} + \frac{[b_{14}(\lambda_2 - a_{11})]\delta^*u^2}{a_{12}(\lambda_2+1)} - \frac{a_{12}b_{24}\delta^*v^2}{a_{12}(\lambda_2+1)},$$

$$g(u,v,\delta^*) = \frac{[a_{13}(1+a_{11}) + a_{12}a_{23}]uv}{a_{12}(\lambda_2+1)} + \frac{[a_{14}(1+a_{11})]u^2}{a_{12}(\lambda_2+1)} + \frac{a_{12}a_{24}v^2}{a_{12}(\lambda_2+1)} + \frac{[b_{11}(1+a_{11}) + a_{12}b_{21}]\delta^*u}{a_{12}(\lambda_2+1)} + \frac{[b_{12}(1+a_{11}) + a_{12}b_{22}]\delta^*v}{a_{12}(\lambda_2+1)} + \frac{[b_{13}(1+a_{11}) + a_{12}b_{23}]\delta^*uv}{a_{12}(\lambda_2+1)} + \frac{[b_{14}(1+a_{11})]\delta^*u^2}{a_{12}(\lambda_2+1)} + \frac{a_{12}b_{24}\delta^*v^2}{a_{12}(\lambda_2+1)},$$

$u = a_{12}(x+y)$, and $v = -(1+a_{11})x + (\lambda_2 - a_{11})y$.

Applying center manifold theorem to eq. (3.4) at the origin in limited neighborhood of $\delta^* = 0$. The center manifold $W^c(0,0)$ can be approximately presented as:

$$W^c(0,0) = \{(\tilde{x}, \tilde{y}): \tilde{y} = a_0\delta^* + a_1\tilde{x}^2 + a_2\tilde{x}\delta^* + a_3{\delta^*}^2 + O((|\tilde{x}| + |\delta^*|)^3)\},$$

where $O((|\tilde{x}| + |\delta^*|)^3)$ is a function with at least third order in variables$(\tilde{x}, \delta^*)$.

By simple calculations for center manifold, we have

$a_0 = 0$,

$$a_1 = \frac{[a_{13}(1+a_{11}) + a_{12}a_{23}](1+a_{11}) - a_{12}a_{14}(1+a_{11}) - a_{24}(1+a_{11})^2}{(\lambda_2+1)(\lambda_2+3)},$$

$$a_2 = \frac{-[b_{11}(1+a_{11}) + a_{12}b_{21}]a_{12} + [b_{12}(1+a_{11}) + a_{12}b_{22}](1+a_{11})}{a_{12}(\lambda_2+1)^2},$$

$a_3 = 0$.

Now, consider the map restricted to the center manifold $W^c(0,0)$ as below:

$h: \tilde{x} \to -\tilde{x} + h_1\tilde{x}^2 + h_2\tilde{x}\delta^* + h_3\tilde{x}^2\delta^* + h_4\tilde{x}\delta^{*2} + h_5\tilde{x}^3 + O((|\tilde{x}| + |\delta^*|)^4)$, (3.5)

where

$h_1 = \frac{1}{(\lambda_2+1)}\{-(1+a_{11})[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}] + a_{12}a_{14}(\lambda_2 - a_{11}) - a_{24}(\lambda_2 - a_{11})^2\}$,

$h_2 = \frac{1}{(\lambda_2+1)}[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}] - \frac{1}{a_{12}(\lambda_2+1)}(1 + a_{11})[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}]$,

$h_3 = \frac{1}{(\lambda_2+1)}\{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}](\lambda_2 - 2a_{11} - 1)a_2 + 2a_{12}a_{14}(\lambda_2 - a_{11})a_2 + a_{24}(1 + a_{11})(\lambda_2 - a_{11})a_2 + [b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]a_1 - [b_{13}(\lambda_2 - a_{11}) - a_{12}b_{23}](1 + a_{11}) + a_{12}b_{14}(\lambda_2 - a_{11}) - b_{24}(1 + a_{11})^2\} + \frac{1}{a_{12}(\lambda_2+1)}[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}](\lambda_2 - a_{11})a_1$,

$h_4 = \frac{1}{(\lambda_2+1)}[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]a_2 + \frac{1}{a_{12}(\lambda_2+1)}(\lambda_2 - a_{11})[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}]a_2$,

$h_5 = \frac{1}{(\lambda_2+1)}\{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}](\lambda_2 - 2a_{11} - 1)a_1 + 2a_{12}a_{14}(\lambda_2 - a_{11})a_1 + a_{24}(1 + a_{11})(\lambda_2 - a_{11})a_1\}$.

According to Flip bifurcation, the discriminatory quantities γ_1 and γ_2 are given by:

$$\gamma_1 = \left(\frac{\partial^2 f}{\partial\tilde{x}\partial\delta^*} + \frac{1}{2}\frac{\partial f}{\partial\delta^*}\frac{\partial^2 f}{\partial\tilde{x}^2}\right)\Big|_{(0,0)},$$

$$\gamma_2 = \left(\frac{1}{6}\frac{\partial^3 f}{\partial\tilde{x}^3} + \left(\frac{1}{2}\frac{\partial^2 f}{\partial\tilde{x}^2}\right)^2\right)\Big|_{(0,0)}.$$

After simple calculations, we obtain $\gamma_1 = h_2$ and $\gamma_2 = h_5 + h_1^2$.

Analyzing above and the flip bifurcation [20], we write a theorem similar to [11] as below:

Theorem 3.1. If $\gamma_2 \neq 0$, and the parameter δ^* alters in the limiting region of the point (0, 0), then the system (3.2) passes through flip bifurcation at the point $B(x^*, y^*)$. Also, the period-2 points that bifurcate from fixed point $B(x^*, y^*)$ are stable (resp., unstable) if $\gamma_2 > 0$ (resp., $\gamma_2 < 0$).

III.2 HOPF BIFURCATION

Consider the system (1.2) with arbitrary parameter $(a_2, b_2, c_2, d_2, l_2, m_2, \delta_2) \in H_B$, which is described as follows:

$$\begin{cases} x \to x + \delta_2[x(a_2 - c_2x) - l_2xy], \\ y \to y + \delta_2[y(-b_2 - d_2y) + m_2xy] \end{cases} \quad (3.6)$$

(3.6) has fixed point $B(x^*, y^*)$, where $x^* = \frac{a_2d_2+b_2l_2}{l_2m_2+c_2d_2}$, $y^* = \frac{a_2m_2-b_2c_2}{l_2m_2+c_2d_2}$ and $\delta_2 = \frac{(a_2c_2d_2+a_2d_2m_2+b_2c_2l_2-b_2c_2d_2)}{(a_2d_2+b_2l_2)(a_2m_2-b_2c_2)}$.

Consider the perturbation of (3.6) as follows:

$$\begin{cases} x \to x + (\delta_2 + \delta)[x(a_2 - c_2x) - l_2xy], \\ y \to y + (\delta_2 + \delta)[y(-b_2 - d_2y) + m_2xy], \end{cases} \quad (3.7)$$

where $|\delta| \ll 1$ is small perturbation parameter.

The characterization equation of map (3.7) at $B(x^*, y^*)$ is given by $\lambda^2 + p(\delta) + q(\delta) = 0$,

where

$$p(\delta) = -2 + \frac{(\delta_2+\delta)(a_2c_2d_2+a_2d_2m_2+b_2c_2l_2-b_2c_2d_2)}{(l_2m_2+c_2d_2)},$$

$$q(\delta) = 1 - \frac{(\delta_2+\delta)(a_2c_2d_2+a_2d_2m_2+b_2c_2l_2-b_2c_2d_2)}{(l_2m_2+c_2d_2)} + \frac{(\delta_2+\delta)^2(a_2d_2+b_2l_2)(a_2m_2-b_2c_2)}{(l_2m_2+c_2d_2)}$$

Since the parameter $(a_2, b_2, c_2, d_2, l_2, m_2, \delta_2) \in H_B$, the eigen values of $B(x^*, y^*)$ are a pair of conjugate complex numbers $\bar{\lambda}$ and λ with modulus 1, where

$$\bar{\lambda}, \lambda = \frac{-p(\delta) \mp i\sqrt{4q(\delta) - p^2(\delta)}}{2}.$$

Now we have

$$|\lambda| = (q(\delta))^{1/2},$$

$$l = \frac{d|\lambda|}{d\delta}\Big|_{\delta=0} = \frac{(a_2c_2d_2+a_2d_2m_2+b_2c_2l_2-b_2c_2d_2)}{2(l_2m_2+c_2d_2)} > 0.$$

When δ varies in small neighborhood of $\delta = 0$, then $\bar{\lambda}, \lambda = \alpha \mp i\beta$.

Hopf bifurcation requires that when $\delta = 0$, $\bar{\lambda}^n, \lambda^n \neq 1$ $(n = 1, 2, 3, 4)$ which is equivalent to $p(0) \neq -2, 0, 1, 2$.

Since the parameter $(a_2, b_2, c_2, d_2, l_2, m_2, \delta_2) \in H_B$, therefore $p(0) \neq -2, 2$. We only require that $p(0) \neq 0, 1$, which leads to

$(a_2c_2d_2 + a_2d_2m_2 + b_2c_2l_2 - b_2c_2d_2)^2 = j(l_2m_2 + c_2d_2)(a_2d_2 + b_2l_2)(a_2m_2 - b_2c_2), j = 2,3.$ (3.8)

Let $u = x - x^*$ and $v = y - y^*$.

After transformation the fixed point $B(x^*, y^*)$ of map (3.7) to the point (0, 0). We have

$u \to u + (\delta_2 + \delta)[a_2u - 2c_2x^*u - l_2y^*u - c_2u^2 - l_2v(u + x^*)]$ and $v \to v + (\delta_2 + \delta)[-b_2v - 2d_2y^*v + m_2x^*v - d_2v^2 + m_2u(v + y^*)$ (3.9)

After that we discuss the normal form of (3.9) when $\delta = 0$.

Consider the following translation:

$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$, where $T = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}$.

Taking T^{-1}on both sides of (3.9), we get

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{f}(\tilde{x}, \tilde{y}) \\ \tilde{g}(\tilde{x}, \tilde{y}) \end{pmatrix}, \quad (3.10)$$

where

$\tilde{f}(\tilde{x}, \tilde{y}) = \frac{\delta_2}{\beta}[\alpha c_2u^2 + (\alpha l_2 + m_2)uv - d_2v^2]$,

$\tilde{g}(\tilde{x}, \tilde{y}) = -\delta_2[c_2u^2 + l_2uv]$,

$u = \tilde{y}$ and $v = \beta\tilde{x} + \alpha\tilde{y}$.

According to Hopf bifurcation, the discriminatory quantity k is given by

$$\boldsymbol{k} = -\boldsymbol{Re}\left[\frac{(1-2\bar{\lambda})\bar{\lambda}^2}{1-\lambda}\boldsymbol{\varphi_{11}\varphi_{20}}\right] - \frac{1}{2}\|\boldsymbol{\varphi_{11}}\|^2 - \|\boldsymbol{\varphi_{02}}\|^2 + \boldsymbol{Re}(\bar{\boldsymbol{\lambda}}\boldsymbol{\varphi_{21}}), \quad (3.11)$$

where

$$\boldsymbol{\varphi_{20}} = \frac{1}{8}[\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}_{\tilde{y}\tilde{y}} + 2\tilde{g}_{\tilde{x}\tilde{y}} + i(\tilde{g}_{\tilde{x}\tilde{x}} - \tilde{g}_{\tilde{y}\tilde{y}} - 2\tilde{f}_{\tilde{x}\tilde{y}})],$$

$$\boldsymbol{\varphi_{11}} = \frac{1}{4}[\tilde{f}_{\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{y}\tilde{y}} + i(\tilde{g}_{\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{x}\tilde{y}})],$$

$$\boldsymbol{\varphi_{02}} = \frac{1}{8}[\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}_{\tilde{y}\tilde{y}} + 2\tilde{g}_{\tilde{x}\tilde{y}} + i(\tilde{g}_{\tilde{x}\tilde{x}} - \tilde{g}_{\tilde{y}\tilde{y}} + 2\tilde{f}_{\tilde{x}\tilde{y}})],$$

$$\boldsymbol{\varphi_{21}} = \frac{1}{16}[\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{x}\tilde{y}\tilde{y}} + \tilde{g}_{\tilde{x}\tilde{x}\tilde{y}} + \tilde{g}_{\tilde{y}\tilde{y}\tilde{y}} + i(\tilde{g}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{x}\tilde{y}\tilde{y}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{y}} - \tilde{f}_{\tilde{y}\tilde{y}\tilde{y}})],$$

On solving (3.11), we obtain the value of $\boldsymbol{k}$.

Analyzing above and Hopf bifurcation [20], we write a theorem similar to [11] as below:

Theorem 3.2. *If the condition (3.8) holds,* $\boldsymbol{k} \neq \boldsymbol{0}$ *and the parameter* $\boldsymbol{\delta}$ *alters in the limited region of the point (0, 0), then the system (3.7) passes through Hopf bifurcation at the point* $\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{y}^*)$*. Moreover, if* $\boldsymbol{k} < \boldsymbol{0}$ *(resp.,* $\boldsymbol{k} > \boldsymbol{0}$*), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point* $\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *for* $\boldsymbol{\delta} > \boldsymbol{0}$ *(resp.,* $\boldsymbol{\delta} < \boldsymbol{0}$*).*

IV. NUMERICAL SIMULATIONS

To verify the theoretical analysis, we draw the bifurcation diagrams, largest Lypunov exponents and phase portraits for the system (1.2). This shows the complete dynamical behavior and the global stability of prey-predator system at the fixed points. We discuss bifurcation in following cases:

Case 1: In this case, we draw the bifurcation diagram of the model (1.2) taking $\boldsymbol{a} = \boldsymbol{3.7}$, $\boldsymbol{b} = \boldsymbol{1}$, $\boldsymbol{c} = \boldsymbol{2}$, $\boldsymbol{d} = \boldsymbol{0.9}$, $\boldsymbol{l} = \boldsymbol{1}, \boldsymbol{m} = \boldsymbol{0.9}$, the initial value of $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{0.86}, \boldsymbol{0.63})$ and $\boldsymbol{\delta}$ covering [0.5, 0.9]. From Fig. 4.1 (a), we see that from the fixed point $(\boldsymbol{1.6037}, \boldsymbol{0.4925})$, flip bifurcation appears at $\boldsymbol{\delta} = \boldsymbol{0.6848}$ having $\boldsymbol{\gamma_1} = -7.1235$ and $\gamma_2 = 18.8$ and $(a, b, c, d, l, m) = (3.7, 1, 2, 0.9, 1, 0.9) \in F_{B1}$. It shows that the Theorem 3.1 is correct.

The phase portraits in the Fig. 4.2 shows that there are chaotic sets at $\delta = 0.79, 0.8$. Moreover, the Largest Lypunov exponents corresponding to $\delta = 0.79, 0.8$ are positive that confirm the chaotic sets.

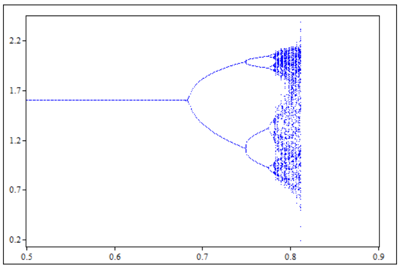


Fig.4.1.(a)

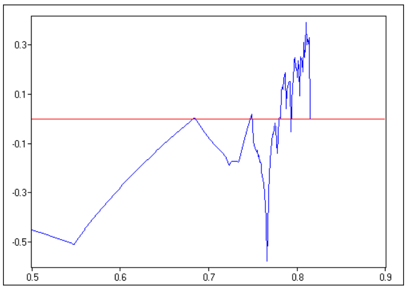
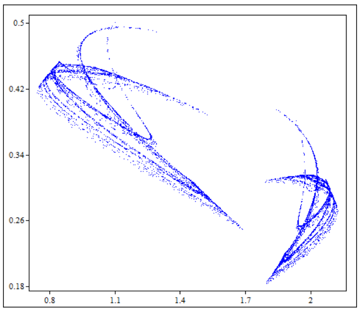
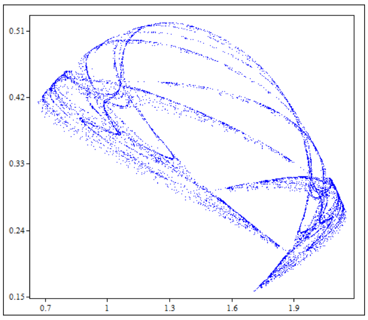


Fig. 4.1. (b)

Fig. 4.1 (a) Bifurcation diagram of system (1.2) with δ covering [0.5, 0.9], a = 3.7, b = 1, c = 2, d = 0.9, l = 1, m = 0.9, the initial value of (x, y) = (0.86, 0.63), where horizontal axis, vertical axis presents δ, x respectively. (b) Largest Lyapunov exponents related to 4.1 (a).



$\delta = 0.790$



$\delta = 0.8$

Fig. 4.2. Phase portraits for several values of δ corresponding to Fig. 4.1 (a) where horizontal axis, vertical axis presents x, y respectively.

Case 2: In this case, we draw the bifurcation diagram of the model (1.2) taking $a = 1$, $b = 0.9$, $c = 0.4$, $d = 0.7$, $l = 0.5, m = 1$, the initial value of $(x, y) = (0.7, 0.3)$ and δ covering [1, 1.4]. We see from Fig. 4.3 (a) that from the fixed point $(1.4743, 0.8205)$, Hopf bifurcation emerges at $\delta = 1.2336$ with $\alpha = 0.28192$, $\beta = 0.95943$, $k = -0.3669$ and $(a, b, c, d, l, m) = (1, 0.9, 0.4, 0.7, 0.5, 1) \in H_B$. It shows that the Theorem 3.2 is accurate.

From the Fig 4.3(a), it has been observed that when $\delta < 1.2336$, the fixed point $(1.4743, 0.8205)$ of the system (1.2) is stable. At $\delta = 1.2336$ the fixed point loses its stability and as δ exceeds from 1.2336, an invariant circle generates.

The largest Lyapunov exponents in Fig. 4.3(b) shows that for the parameter $\delta \in (1, 1.2336)$, the Lyapunov exponents are negative. For $\delta \in (1.2336, 1.34200)$, some Lyapunov exponents are positive and some negative, it means that there exist stable period windows in chaotic region.

The phase portraits in Fig. 4.4, shows that a smooth invariant circle bifurcates from the fixed point $(1.4743, 0.8205)$. There appears an invariant circle for some values of δ and its radius becomes larger with the growth of δ e.g. when δ varies from 1.2336 to 1.23787. There appears a closed curve for some values of δ and its shape changes with the growth of δ e.g. when δ varies from 1.25123 to 1.342.

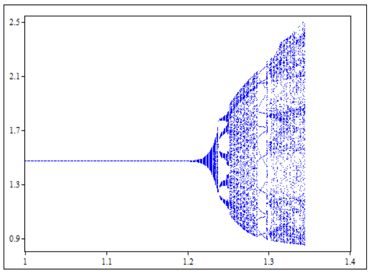


Fig. 4.3(a)

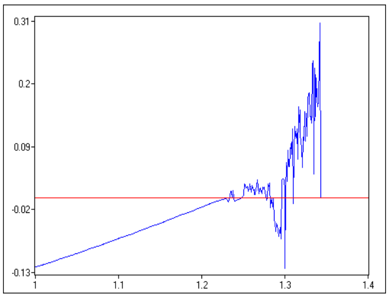
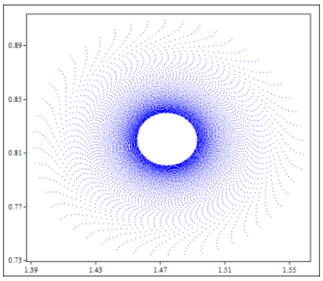
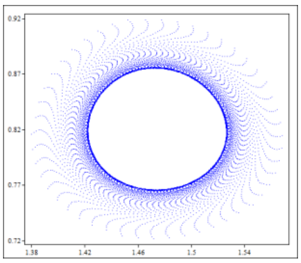


Fig. 4.3(b)

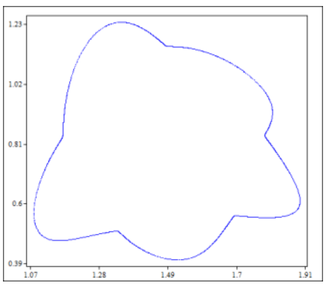
Fig. 4.3(a) Bifurcation diagram of system (1.2) with δ covering [1, 1.4], a = 1, b = 0.9, c = 0.4, d = 0.7, l = 0.5, m = 1 the initial value of (x, y) = (0.7, 0.3), where horizontal axis, vertical axis presents δ, x respectively. (b) Largest Lyapunov exponents related to (a).



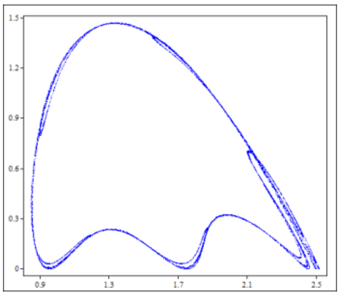
δ = 1.2336



δ = 1.2378



δ = 1.25123



δ = 1.342

Fig. 4.4. Phase portraits for several values of δ related to Fig. 4.3 (a) where horizontal axis, vertical axis presents x, y respectively.

V. CONCLUSIONS

In this paper, we investigated the dynamical behaviors of discrete-time prey-predator model in the closed first quadrant R_+^2. Global stability of the model at the fixed points has been discussed. The map undergoes flip bifurcation and Hopf bifurcation at the fixed point under specific conditions, when δ varies in small neighbourhood of F_{B1}or F_{B2} and H_B . Numerical simulations display cascade of period doubling bifurcation, chaotic sets in case of flip bifurcation, and smooth invariant circle and closed curves in case of Hopf bifurcation. The complexity of dynamical behaviors is confirmed by computation of Lyapunov exponents.

Further, Liu et al. [19] for corresponding continuous predator-prey model concluded that the system is globally asymptotically stable under certain conditions, whereas we observed that the discrete system has a rich and complex dynamical behavior than the continuous system.

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