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Exact Finite Element for Torsional Vibration of Shafts under Harmonic Torsion

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***Abstract*— The present paper investigate the dynamic analysis for the torsional vibration of shafts subjected to various harmonic twisting moments. The governing torsional vibration equation and related boundary conditions for the shafts are derived using Hamilton's variational principle. The exact closed-form solutions for the shafts having a cantilevered and simply supported boundary conditions and under various harmonic torsional loads are determined. The closed form solution is consequently used to develop a family of exact shape functions which exactly satisfy the exact homogeneous solution of the governing torsional equation. A super-convergent two-noded finite beam element is then formulated based on the exact shape functions. The proposed beam element developed involves no special discretization errors normally encountered in conventional finite element formulations and provide results in excellent agreement with a minimal number of degrees of freedom. The present solutions are shown to successfully capture the static and steady state dynamic responses of shafts. Both solutions are also able to predict the natural torsional frequencies and related mode-shapes of the shafts. The validity and the accuracy of the present exact closed-form and finite beam element solutions are achieved throughout the numerical examples presented and compared with well-established ABAQUS finite beam solution and other exact solution.**

Keywords*— Exact finite element, closed form solution, torsional response, harmonic torsional loading, steady state response

I. INTRODUCTION AND OBJECTIVE

Shafts and beams of circular cross-sections are important structural elements of rotating and non-rotating machines and systems, mainly used to transmit torque and rotation. These shafts are often idealized as simple rotating and non-rotating beam models such as helicopter blades, aircraft rotary wings and gas turbine blades, etc. In such applications, these shafts are subjected to dynamic torsional loadings caused by machinery, environmental shocks, and dynamic exciting torque as repeated harmonic twisting moments in the long range duration, Therefore, the torsional vibration of shafts under harmonic torsional cyclic loading produce stresses reversal that affect the structural machine integrity and the life of components and then are prone to fatigue failure. Under harmonic twisting moments, the steady state dynamic component of the total response is sustained for a long time and is thus of great importance in fatigue design of such shafts. While the transient component of total response which is introduced only at the beginning of the excitation tends to dampen out quickly and is thus of a very little importance in assessing the fatigue life of the shaft. Thus, the objective of this paper is to develop an exact closed form solution and accurate and efficient finite element solution which captures and isolates the steady state torsional dynamic response of shafts subjected to general harmonic twisting moments. The present solutions developed in this study are also capable accurately to capture the quasi-static response and predict the eigen-frequencies and eigen-modes of the shaft.

II. LITERATURE REVIEW

Several studies have been conducted on the exact solution and finite element for torsional vibration analysis of shafts under various dynamic loads and boundary conditions. Among them, [1] derived the exact expressions for torsional frequencies and mode shapes for the free torsional vibrations of circular shafts and piping systems constrained by unsymmetrical torsional springs and carrying multiple unequal rotational masses at arbitrary locations on the shaft. Reference [2] developed a boundary element solution for the general linear elastic non-uniform torsion problem of homogeneous or composite prismatic bars of arbitrary cross section subjected to various twisting moment and linear torsional boundary conditions. In machinery systems, torsional vibration problems are important research topic of rotor-shaft systems. Therefore, a lot of researchers have studied the free vibration frequencies of rotating shafts and beams. Rotating shaft is differed from the non-rotating one in having an additional centrifugal force and Coriolis effects on its dynamic behavior. For instance, [3] derived the finite element stiffness and mass matrices for both open and closed section shafts, based on the solution of the static equations, and thus obtained an approximate solution for the natural frequencies. Reference [4] introduced the classical Euler-Bernoulli beam theory to present an analytical solution to investigate the vibrational behavior of a beam rotating with constant speed about its longitudinal axis for all boundary conditions. Reference [5] presented an analytical solution for free flexural vibration of a spinning finite Timoshenko beam subjected to a moving load for the general boundary conditions. Their formulation accounts for the rotary inertia and shear deformation effects**.** Reference [6] studied the dynamic response of a rotating shaft subject to axial force and moving loads is analyzed by using Timoshenko beam theory and the assumed mode method. Reference [7] presented an analytical approach to study the torsional vibrations of the drive shafts and mechanisms. Reference [8] studied the free vibrations and stability of internally damped rotating shafts with general boundary conditions. Reference [9] derived the exact governing equations for linear vibration of a rotating Timoshenko beam by using d'Alembert principle and the virtual work principle. Based on the assumptions that the beam is linear elastic and the steady state axial strain is small, the formulation is captured the effect of Coriolis force on the natural frequency of the rotating Timoshenko beam**.** Reference [10] investigated the free vibrations analysis of a rotating shaft with nonlinearities in curvature and inertia.

Moreover, the finite element method is used for the torsional vibration analysis of solid and hollow shafts and beams in determining the natural frequencies and mode shapes. This includes the work of [11] developed a finite element solution for determining the free vibration characteristics of rotating uniform Timoshenko beams. His formulation incorporated the effects of shear deformation and rotary inertia on the natural frequencies of the rotating beams. Reference [12] derived explicit expressions for the finite element mass and stiffness matrices using consistent mass formulation for the vibration of a rotating tapered beam. Reference [13] developed a three nodal $C°$ Timoshenko finite beam element to evaluate the natural whirling speeds of a rotating shaft with various end boundary conditions. Reference [14] presented a simple spinning composite shaft model based on a first-order shear deformable beam theory. The finite element model is used to analyze the critical speeds, natural frequencies and related mode shapes of the composite shaft system. A feature common to the above finite element studies is use of approximate shape functions involving spatial discretization errors, and thus requiring fine meshes to converge to the actual solution. In contrast, the present study avoids discretization errors by formulating exact shape functions which exactly satisfy the homogeneous solution of the dynamic governing torsional equation. Another commonality between the above studies is the fact they focus on extracting the free vibration characteristics including predicting the natural frequencies and mode shapes. In contrast, the present solution aims at directly extracting the steady state torsional dynamic response without the need to extracting the natural frequencies and mode shapes.

III. KINEMATICS FUNCTIONS

A straight uniform circular closed cross-section beam of length L has is shown in Fig. (1). The shaft is referenced to a right-handed rectangular coordinates system (X,Y,Z), where the axis X is the longitudinal axis of the beam, Y and Z are the principal axes of the cross-section. The present theoretical formulation is based on the following main assumptions:

1. The formulation is applicable to shafts having circular solid/hollow cross-sections,
2. Cross-section is assumed to remain perfectly rigid in its own plane throughout deformation,
3. The material is assumed to remain linearly elastic throughout deformation,
4. Displacements, strains and rotations are assumed small, and
5. Damping is neglected in the formulation.
6. The formulation is captured only the steady state dynamic response.

According to the assumptions described above, the displacement functions of an arbitrary point $p(x,y,z)$ located on the shaft cross-section, as shown in Figure (1), can be expressed as:

$$u_p(x,t)=0 \tag{1}$$

$$v_p(x,t)=-r\theta\sin\beta=-z\theta(x,t) \tag{2}$$

$$w_p(x,t)=r\theta\cos\beta=y\theta(x,t) \tag{3}$$

in which $u_p(x,t)$, $v_p(x,t)$ and $w_p(x,t)$ are the longitudinal, lateral and transverse displacements of point $p(x,y,z)$ along the principle axes (X,Y,Z), $\theta(x,t)$ is the torsional displacement, and y,z are the coordinates of point p along the principal axes.

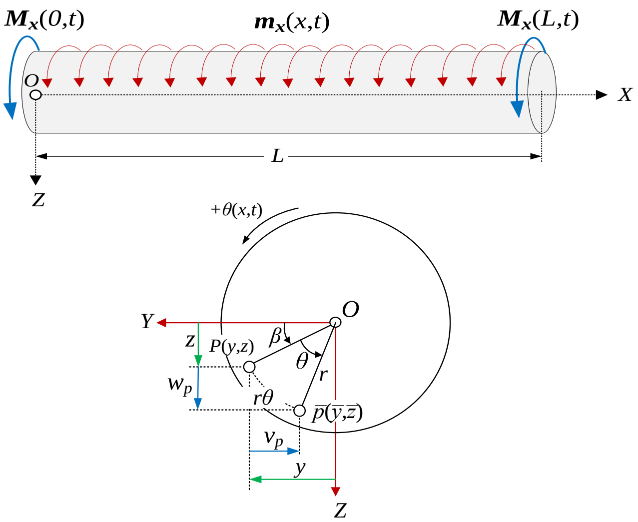

Fig.1: Coordinate system and displacements

Strain-Displacement Relations

Based on the assumption of small displacements, the non-zero shear strains are given by:

$$\gamma_{xy}=\frac{\partial v_p}{\partial x}+\frac{\partial u_p}{\partial y}=-z\frac{\partial\theta}{\partial x}=-z\theta' \tag{4}$$

$$\gamma_{xz}=\frac{\partial w_p}{\partial x}+\frac{\partial u_p}{\partial z}=y\frac{\partial\theta}{\partial x}=y\theta' \tag{5}$$

Torsional Displacement Function

The shaft is assumed to be subjected to harmonic torsion; (i) distributed twisting moment $m_x(x,t)=\bar{m}_x(x)\,e^{i\Omega t}$ acting along the shaft axis, and (ii) concentrated twisting moments $M(x_e,t)=\bar{M}_x(x_e)\,e^{i\Omega t}$ at the shaft both ends (i.e., $e=0,L$), as shown in Fig. (1). The harmonic twisting moments can be written by:

$$m_x(x,t),M_x(x_e,t)=\left[\bar{m}_x(x),\bar{M}_x(x_e)\right]e^{i\Omega t} \tag{6}$$

where Ω is the frequency of the applied twisting moments, $i=\sqrt{-1}$ is the value of the imaginary, $m_x(x,t)$ is the harmonic distributed torsion, $M_x(x_e,t)$ is the concentrated harmonic twisting moment applied at both ends of the given shaft. Under the effect of the applied harmonic twisting moments, the angular displacement $\theta(x,t)$ is assumed to be harmonic, i.e.,

$$\theta(x,t)=\Theta_x(x)e^{i\Omega t} \tag{7}$$

in which $\Theta_x(x)$ represents the space function of the steady state torsional response. In line with the main objective of this paper focusing on steady state response, the torsional displacement function postulated in equation (7) neglect the transient component of the response.

IV. TORSIONAL VIBRATION EQUATION

The variational form of the Hamiltonian functional δH is taken to be stationary, i.e.,

$$\delta H = \int_{t_1}^{t_2} \delta(T - U + W)\,dt = 0 \qquad for \;\; \delta\theta(x,t_1) = \delta\theta(x,t_2) = 0 \tag{8}$$

in which the integration is performed between arbitrary time limits t_1 and t_2, where δT is the variation of kinetic energy, defined by:

$$\delta T = \int_0^L \int_A \rho\left[\dot{u}_p \delta\dot{u}_p + \dot{v}_p \delta\dot{v}_p + \dot{w}_p \delta\dot{w}_p\right] dA dx \tag{9}$$

δU is the variation of internal strain energy, defined by:

$$\delta U = \int_0^L \int_A \left[G\gamma_{xy}\delta\gamma_{xy} + G\gamma_{xz}\delta\gamma_{xz}\right] dA dx \tag{10}$$

and δW is the variation of the virtual work done for the shaft subjected to given applied torsions, given as:

$$\delta W = \int_0^L m_x(x,t)\delta\theta(x,t)dx + \left[M_x(x,t)\delta\theta(x,t)\right]_{x=x_e} \tag{11}$$

where ρ is the density of the shaft material, and G is the modulus of rigidity of the shaft.

From Equations (1-7) and by substituting into energy expressions (9-11), the resulting equations into Hamilton's equation (8), performing integration by parts, the governing equation for torsional vibration and related boundary conditions of shafts under harmonic twisting moments are given in terms of variable x as:

$$GJ\,\Theta_x''(x) + \rho J \Omega^2 \Theta_x(x) = -\bar{m}_x(x) \tag{12}$$

and the boundary conditions are:

$$\delta\Theta_x(x)\Big|_0^L = 0 \;\; \text{or} \;\; GJ\,\Theta_x'(x)\Big|_0^L = \bar{M}_x(x)\Big|_0^L \tag{13}$$

where J is the torsional coefficient defined by;

$J = I_z + I_y = \int_A (y^2 + z^2)\,dA$.

V. GENERAL EXACT CLOSED FORM SOLUTION

The general exact closed form solution $\Theta_x(x)$ of equation (12) consists of two parts, homogeneous solution $\Theta_{xh}(x)$ and particular solution $\Theta_{xp}(x)$.

Exact Homogeneous Solution for Torsional Displacement

The exact homogeneous solution of the torsional equation is obtained by (a) setting the right hand side of the equation (12) equal to zero, i.e., $\bar{m}_x(x) = 0$, and (b) assuming the torsional displacement to take the following exponential form:

$$\Theta_{xh}(x) = A_i e^{m_i x} \tag{14}$$

From equation (14) by substituting into homogeneous form of equation (12), the exact homogeneous solution for torsional displacement is written in matrix form as:

$$\Theta_{xh}(x) = \left\langle e^{m_1 x} \quad e^{m_2 x}\right\rangle_{1\times2} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_{2\times1} = \left\langle E(x)\right\rangle_{1\times2}\left\{\bar{A}\right\}_{2\times1} \tag{15}$$

where $m_{1,2} = \pm i\Omega\sqrt{\rho/G}$, $\left\langle E(x)\right\rangle_{1\times2} = \left\langle e^{m_1 x} \quad e^{m_2 x}\right\rangle_{1\times2}$, $\left\langle \bar{A}\right\rangle_{1\times2} = \left\langle A_1 \quad A_2\right\rangle_{1\times2}$ are unknown integration constants. Equation (15) governs the steady state dynamic solution for torsional vibration of shafts under harmonic twisting moments.

Particular Solution for Torsional Displacement

For a shaft subjected to harmonic distributed twisting moment $\bar{m}_x(x)\,e^{i\Omega t} = \bar{m}_x\,e^{i\Omega t}$, the corresponding particular solution $\Theta_p(x)$ for the governing torsional vibration equation (12) is obtained by:

$$\Theta_{xp}(x) = -\bar{m}_x / \rho J \Omega^2 \tag{16}$$

The complete steady state exact solution for torsional vibration response is obtained by adding the homogeneous solution (15) to the particular solution (16), gives:

$$\Theta_x(x) = \left\langle e^{m_1 x} \quad e^{m_2 x}\right\rangle_{1\times2}\left\{\bar{A}\right\}_{2\times1} + \left(\frac{-\bar{m}_x}{\rho J \Omega^2}\right) \tag{17}$$

Equation (17) represents the complete steady state solution for torsional vibration of shafts under harmonic torsional loading, where the integration constants $\left\{\bar{A}\right\}_{2\times1}$ can be determined from the relevant boundary conditions of the problem.

Exact Solutions for Cantilevered and Simply-supported Shafts under Harmonic Torsion

Shafts having cantilevered and simply-supported boundary conditions subjected to harmonic twisting moments are shown in Fig (2). The boundary conditions of the cantilevered shaft at both ends are given as:

$$\delta\Theta_x(x)\Big|_{x=0} = 0 \;\; \text{and} \;\; GJ\,\Theta_x'(x)\Big|_{x=L} = \bar{M}_x(x)\Big|_{x=L}$$

while for simply-supported shaft are given by:

$\delta\Theta_x(x)\Big|_{x=0} = \delta\Theta_x(x)\Big|_{L=0} = 0$.

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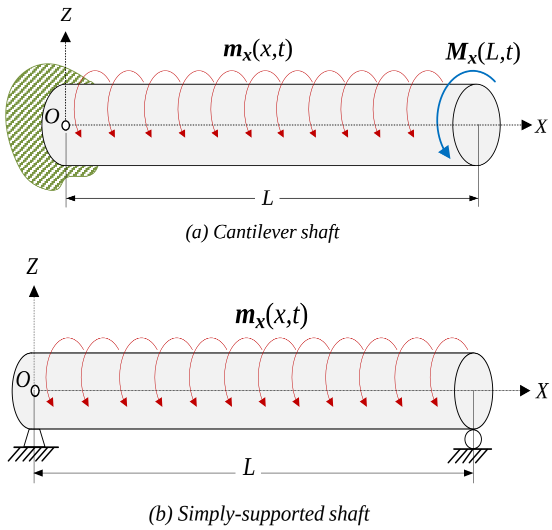

Fig. (2): Shafts with cantilevered and simply-supported boundary conditions under harmonic torsions

Substituting the above boundary conditions into equation (17), the general exact closed form solutions governing the torsional steady state dynamic responses for cantilevered and simply-supported shafts under harmonic twisting moments are represented, respectively, by the following relations:

$$\Theta_{xc}(x)=\left\langle e^{m_1 x}\quad e^{m_2 x}\right\rangle_{1\times2}\begin{bmatrix}1 & 1\\ m_1 e^{m_1 L} & m_2 e^{m_2 L}\end{bmatrix}_{2\times2}^{-1}\begin{Bmatrix}\dfrac{\bar{m}_x}{\rho J\Omega^2}\\ \dfrac{\bar{M}_x(L)}{GJ}\end{Bmatrix}_{2\times1}+\left(\frac{-\bar{m}_x}{\rho J\Omega^2}\right) \tag{18}$$

$$\Theta_{xs}(x)=\left\langle e^{m_1 x}\quad e^{m_2 x}\right\rangle_{1\times2}\begin{bmatrix}1 & 1\\ e^{m_1 L} & e^{m_2 L}\end{bmatrix}_{2\times2}^{-1}\begin{Bmatrix}\dfrac{\bar{m}_x}{\rho J\Omega^2}\\ \dfrac{\bar{m}_x}{\rho J\Omega^2}\end{Bmatrix}_{2\times1}+\left(\frac{-\bar{m}_x}{\rho J\Omega^2}\right) \tag{19}$$

VI. FINITE ELEMENT FORMULATION

In this section, a new two-noded finite beam element is developed for dynamic torsional analysis of structural members with circular cross-sections under various harmonic twisting moments. Figure (3) shows the proposed two-noded finite beam element with two degrees of freedom per element. A family of exact shape functions which exactly satisfy the homogeneous solution of the dynamic torsional equation is employed to formulate the exact stiffness and mass matrices and the load potential vector.

Formulating Exact Torsional Shape Functions

To relate the exact homogeneous solution of the torsional displacement function $\Theta_{xh}(x)$ to the nodal torsional displacements $\{\Phi_N\}_{2\times1}$, the vector of integration constants $\{\bar{A}\}_{2\times1}$ is expressed in terms of the nodal torsional displacements $\langle\Phi_N\rangle_{1\times2}=\langle\varphi_{x1}\quad\varphi_{x2}\rangle_{1\times2}$ by imposing the conditions $\Theta_{xh}(0)=\varphi_{x1}$ and $\Theta_{xh}(L)=\varphi_{x2}$ (Fig. 3), yielding:

$$\begin{aligned}\{\Phi_N\}_{2\times1}&=\begin{Bmatrix}\Theta_{xh}(0)\\ \Theta_{xh}(L)\end{Bmatrix}_{2\times1}\\ &=\begin{bmatrix}\langle E(0)\rangle_{1\times2}\\ \langle E(L)\rangle_{1\times2}\end{bmatrix}_{2\times2}\{\bar{A}\}_{2\times1}=[G_\alpha]_{2\times2}\{\bar{A}\}_{2\times1}\end{aligned} \tag{20}$$

where $\langle\Phi_N\rangle_{1\times2}=\langle\varphi_{x1}\quad\varphi_{x2}\rangle_{1\times2}$ is the vector of the nodal torsional displacements.

From equation (20) and by substituting into equation (17), the following expression is obtained as:

$$\Theta_{xh}(x)=\langle H(x)\rangle_{1\times2}\{\Phi_N\}_{2\times1} \tag{21}$$

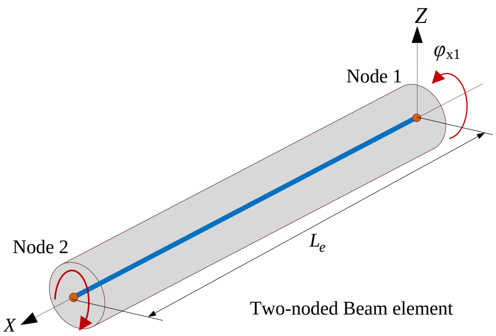

Fig. (3): A proposed two-noded finite beam element

in which $\langle H(x)\rangle_{1\times2}=\langle E(x)\rangle_{1\times2}[G_\alpha]_{2\times2}^{-1}$ is the matrix of exact shape functions for the steady state dynamic torsional response. It is noted that the exact shape functions obtained in equation (21) exactly satisfy the exact homogeneous solution of the governing torsional equation. The exact shape functions developed in this study are depends on the length of the beam element, the properties of the cross-section and the exciting frequency of the torsional loading.

Matrix Formulation for Torsional Response

By using equations (1-7) with torsional displacement equation (15), and by substituting into the energy expressions (9-11), the following variations of energy expressions are given in terms of nodal torsional displacement as:

$$\delta T=-\Omega^2\int_0^L\rho J\langle\Phi_N\rangle_{1\times2}\{H(x)\}_{2\times1}\langle H(x)\rangle_{1\times2}\{\Phi_N\}_{2\times1}dx \tag{22}$$

$$\delta U=\int_0^L GJ\langle\Phi_N\rangle_{1\times2}\{H'(x)\}_{2\times1}\langle H'(x)\rangle_{1\times2}\{\Phi_N\}_{2\times1}dx \tag{23}$$

$$\delta W=\int_0^L\bar{m}_x(x)\langle\Phi_N\rangle_{1\times2}\{H(x)\}_{2\times1}dx-\left[\bar{M}_x(x)\langle\Phi_N\rangle_{1\times2}\{H(x)\}_{2\times1}\right]_0^L \tag{24}$$

From equations (22-24), by substituting into variational form of Hamilton's principle equation (8), leads to typical finite beam element model as:

$$\left([K_e]_{2\times2} - \Omega^2 [M_e]_{2\times2}\right)\{\Phi_N\}_{2\times1} = \{F_e\}_{2\times1} \qquad (25)$$

where the beam element stiffness matrix $[K_e]_{2\times2}$ is given by:

$$[K_e]_{2\times2} = \int_0^L GJ\{H'(x)\}_{2\times1}\langle H'(x)\rangle_{1\times2}\,dx \qquad (26)$$

The mass stiffness matrix $[M_e]_{2\times2}$ of the beam element is:

$$[M_e]_{2\times2} = \int_0^L \Omega^2 \rho J\{H(x)\}_{2\times1}\langle H(x)\rangle_{1\times2}\,dx \qquad (27)$$

The potential energy load vector $\{F_e\}_{2\times1}$ of the element is:

$$\{F_e\}_{2\times1} = \int_0^L \bar{m}_x(x)\{H(x)\}_{2\times1}dx - \left[\bar{M}_x(x)\{H(x)\}_{2\times1}\right]_0^L \qquad (28)$$

The above expressions for stiffness, mass and potential load vector formulated for two-noded finite beam element using the exact torsional shape functions developed to investigate the steady state dynamic torsional response of shafts under various harmonic torsional loadings.

VII. NUMERICAL EXAMPLES AND DISCUSSIONS

The present solutions (exact closed form and finite beam element solutions) developed in this paper governing the torsional vibration of shafts under various harmonic twisting moments can be used to capture the following analyses:

- The steady state dynamic response for the shafts with solid and hollow cross-sections and under harmonic twisting moment at exciting frequency Ω,
- The quasi-static response for the shafts under harmonic twisting moments by using very low exciting frequency compared to the first torsional natural frequency of the shaft,
- Extracting the natural torsional frequencies and related mode shapes of the shaft under harmonic twisting moments.

To demonstrate the validity, accuracy and applicability of the exact closed form solution and finite beam element formulation developed, three examples are presented in this study. These examples investigate the torsional vibration of the shaft under various harmonic twisting moments and various boundary conditions. The present finite beam element developed is based on the exact torsional shapes functions which exactly satisfy the homogeneous form of the governing torsional vibration equation of the shaft. Due to this treatment, the mesh discretization errors induced in the classical finite element solutions using polynomial interpolation shape functions are eliminated. As a result, it is observed that, the results obtained based on a single finite beam element exactly matched with the corresponding results based on the exact closed-form solutions developed in this study up to five significant digits. The numerical nodal results obtained from the present finite beam element are compared with the established finite beam element Abaqus and exact solutions available in the literature. In the finite element Abaqus model, a two-noded B31 beam element having six degrees of freedom per element (i.e., three translations and three rotations) is used for comparison.

Example (1): Cantilever Hollow Shaft under Concentrated Harmonic Twisting Moment

A 3000mm cantilever hollow shaft subjected to a concentrated harmonic twisting moment $M_x = 12.0e^{i\Omega t}\ kNm$ applied at the free end of the cantilever is shown in Fig. (4). The shaft has an outer diameter of 100mm and inner diameter of 80mm and made of steel material with the following properties: the modulus of elasticity $E = 200GPa$, shear modulus $G = 70GPa$ and the density $\rho = 7800kg/m^3$. The purpose of this example is to assess the accuracy and validity of the results obtained from the present finite element formulation. It is required to investigate the following:

(1) The static torsional response of the shaft using very low exciting frequency $\Omega \approx 0.01\omega_1$, where the first natural frequency of the given shaft is $\omega_1 = 261.7Hz$,

(2) The dynamic response of the shaft under harmonic torsion at exciting frequency $\Omega = 1.40\omega_1$.

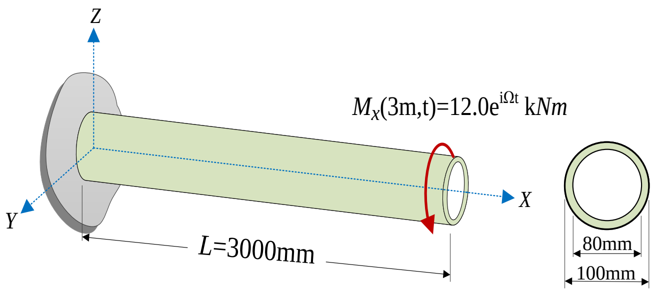

Fig. (4): A cantilever hollow shaft under end harmonic twisting moment

In the Abaqus finite element model, the shaft is divided into 80 beam B31 element along the longitudinal axis of the shaft. In other words, the model has 486 degrees of freedom in order to achieve the required accuracy in this example. In constraint, the results obtained from the present finite element developed use only one two-noded beam element with two degree of freedom to attain the exact solution results.

Static Torsional Response

In order to approach the static response, the exciting frequency should be taken significantly lower than the first natural frequency of the cantilever hollow shaft. The static results for maximum torsional displacement $\varphi_{xmax} (= \varphi_{x2})$ of the shaft under harmonic torsional loading at exciting frequency $\Omega \approx 0.01\omega_1$ are given in Table I. It is observed that the results for the nodal torsional displacement φ_{x2} obtained from the present finite element based on one two-noded beam element (i.e., 2 dof) provide excellent agreement with Abaqus finite element model based on 80 beam B31 elements (i.e., 486 dof). The present finite element formulation based on exact shape functions demonstrated that the new two-noded beam element provides excellent agreement with present exact closed form solution and Abaqus beam model by keeping the number of degrees of freedom a minimum.

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Dynamic Torsional Response

The maximum steady state torsional displacement φ_2 (at the cantilever free end) of the cantilever shaft under the given harmonic twisting moment $M_x = 12.0 e^{i\Omega t}\ kNm$ with exciting frequency $\Omega = 1.40\omega_1 = 366.4Hz$ is provided in Table I. The maximum torsional displacement results based on the formulations developed in this study are compared with those based on Abaqus finite beam B31 element solution. The torsional displacement results obtained from the present exact solution and finite element formulation using a single beam element with 2 dof are found exactly identical to Abaqus beam model based on 80 beam elements with 486 dof in order to achieve the solution accuracy.

Table I: Static and dynamic results for cantilever hollow shaft under end harmonic torsion

TYPE OF RESPONSE	Variable	Present FE (2 dof)	Abaqus FE (486 dof)	Exact Solution
Static $\Omega \approx 0.01\omega_1$	φ_{xmax} (10^{-3} rad)	80.74	80.75	80.74
Dynamic $\Omega = 1.40\omega_1$	φ_{xmax} (10^{-3}rad)	-50.53	-50.53	-50.53

Example (2): Cantilever Shaft under Distributed Torsion

A 4000mm cantilever shaft having solid cross-section of radius 40mm subjected to uniform distributed harmonic twisting moment $m(x,t) = 4000\, e^{i\Omega t}\ Nm$ is considered as shown in Fig. 5. The material of the cantilever is made from steel with the following mechanical properties; modulus of elasticity $E = 200GPa$, modulus of rigidity $G = 70GPa$, and the material density $\rho = 7850kg/m^3$. The objective of this example is to:

(1) Extract the natural torsional frequencies from the dynamic response of the cantilever shaft under the given harmonic torsion, and

(2) Determine the steady state torsional mode shapes of the cantilever shaft corresponded to the natural torsional frequencies.

The cantilever shaft in this example is modelled in Abaqus finite element using 100 beam B31 elements with six degrees of freedom per node along the longitudinal axis of the shaft, i.e., a total of 606 degrees of freedom is used in order to yield the accuracy of this problem. While the present finite element uses a single two-noded beam element with 2 dof to approach the corresponding results obtained from exact closed form solution.

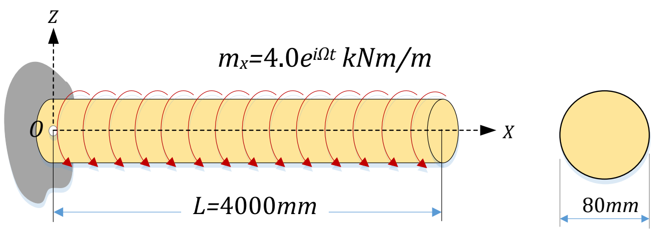

Fig. 5: A cantilever shaft under harmonic distributed twisting moment

Extracting of Natural Torsional Frequencies

Under the given harmonic torsional loading, the natural torsional frequencies are extracted from the multiple steady state torsional dynamic analyses in which the exciting frequency Ω of the torsional loading varying from nearly zero to 1600Hz. Fig. 6 shows the results for the nodal (maximum) torsional displacement φ_2 at the cantilever end against the forcing frequency Ω. The torsional natural frequencies are determined at the peaks of the torsional displacement-frequency diagram as observed in Fig. 6. Peaks on the diagram indicate the resonance and are then identify the torsional natural frequencies of the given cantilever shaft.

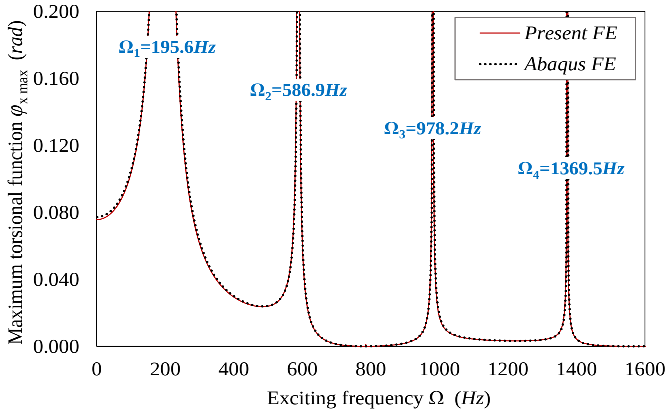

Fig. 6: Natural torsional frequencies of the cantilever under harmonic torsion

The first four torsional natural frequencies extracted from the steady state dynamic responses at the peaks are given in Table II. The values of the torsional natural frequencies obtained from the finite beam element developed in this study using a single two-noded beam element (2 dof) are presented and compared with the corresponding results obtained from the exact closed form solution developed and Abaqus solution using 100 beam B31 element (606 dof). It is noted from the results that, the present finite beam element solution with a minimum degrees of freedom exhibit excellent agreement when compared with other solutions, the exact closed-form solution and Abaqus finite beam B31 element model having a large number of degrees of freedom. Thus, the present finite beam element model is able to capture the eigen-frequencies of the given cantilever shaft.

Table II: Natural torsional frequencies (in *Hz*) for the cantilever shaft under harmonic twisting moment

Number	Exact Solution	Abaqus FE (606 dof)	Present FE (2 dof)
1	195.6	195.6	195.6
2	586.9	586.7	586.9
3	978.2	978.0	978.2
4	1369.6	1369.3	1369.5

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Steady State Torsional Mode Shapes

The first five steady state torsional mode shapes for the dynamic vibration response of the cantilever shaft under the given distributed harmonic torsion $m_x(x,t) = 4000 e^{i\Omega t} Nm$ is shown in Fig. 6. The normalized steady state torsional modes $(\varphi_2/\varphi_{x\max})$ obtained using the present finite beam element are plotted for the first four torsional exciting frequencies: $\Omega_1 = 195.6 Hz$, $\Omega_2 = 586.9 Hz$, $\Omega_3 = 978.2 Hz$ and $\Omega_4 = 1369.5 Hz$, respectively.

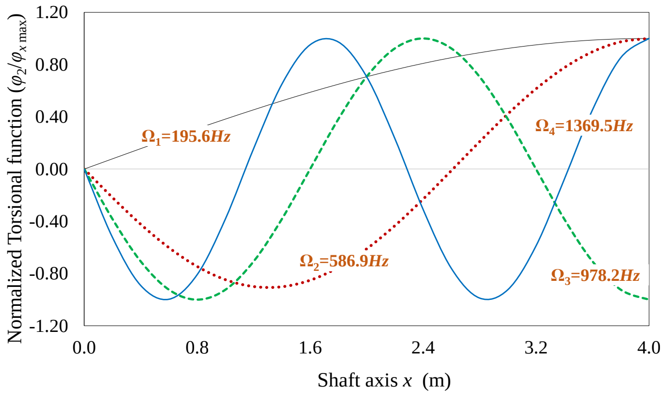

Fig. 7: Normalized torsional mode shapes for cantilever shaft under harmonic distributed torsion

Example (3): Clamped-Clamped Shaft – Verification of The Present Finite Beam Element

This example is presented to exhibit the ability of the present finite element developed in the paper to achieve the required accuracy for the static and dynamic responses for the given problem by comparing the present results with those based on the Abaqus finite beam model solution. A clamped-clamped shaft under various concentrated and distributed harmonic twisting moments is considered as shown in Fig. 8. The shaft has a circular solid cross-section of radius $50mm$ and 8000mm span, while the mechanical material properties are given as: $E = 210\, GPa$, $G = 72\, GPa$ and $\rho = 7850 kg/m^3$. It is required to assess the efficiency and accuracy of the present finite element formulation in evaluating the nodal torsional degrees of freedom $(\varphi_{xi}$ for $i = 1,2,3,4,5)$ for the following analyses:

(i) The static torsional analysis of the clamped-clamped shaft under the given harmonic twisting moments at very low exciting frequency in order to capture the static response.

(ii) The torsional dynamic analysis of the clamped-clamped shaft under the given harmonic twisting moments at exciting frequency $\Omega = 50 Hz$.

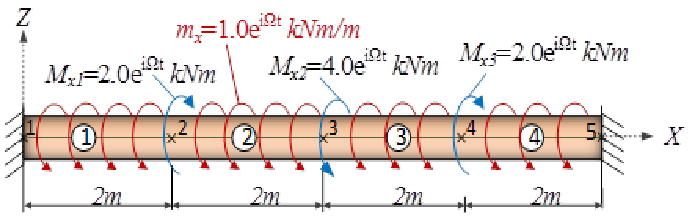

Fig. 7: A clamped-clamped shaft under various harmonic twisting moments

In order to establish the validity and accuracy of the present finite element based on two-noded beam element, the nodal degrees of freedom results for static torsional response and steady state torsional dynamic response are obtained and compared against the results based on established Abaqus finite beam element. Under the present finite element solution, only four two-noded beam elements with a total of 5 degrees of freedom are used while in Abaqus finite element model, the shaft is consisted of two-hundred two-noded beam B31 elements with a total of 1206 degrees of freedom along the shaft longitudinal axis to achieve the convergence.

Static Torsional Response

The static results for the nodal torsional displacement φ_{xi} (for $i = 1,2,3,4,5$) are plotted against the shaft longitudinal axis x as illustrated in Fig. 8a, in which the static torsional response is approached by using a very small exciting frequency Ω. The static results shows that, the nodal torsional degrees of freedom obtained from the present finite beam element formulation having 5 degrees of freedom coincide on the corresponding results obtained from Abaqus beam model having 1206 dof and then provide an excellent agreement.

Dynamic Torsional Response

Fig. 7b shows the nodal degrees of freedom results for the steady state torsional response plotted against the shaft coordinate axis x. It is observed that, the developed finite beam element results based on four two-noded beam elements with 5 degrees of freedom shows again an excellent agreement with those results based on Abaqus finite element model using 200 beam B31 elements with 1206 degrees of freedom. This is a natural outcome of the fact that the present finite beam element formulation is based on the exact shape functions which exactly satisfy the homogeneous solution of the governing torsional vibration equation. This treatment eliminates the discretization errors occurred in the conventional finite element formulations which based on approximate interpolation shape functions.

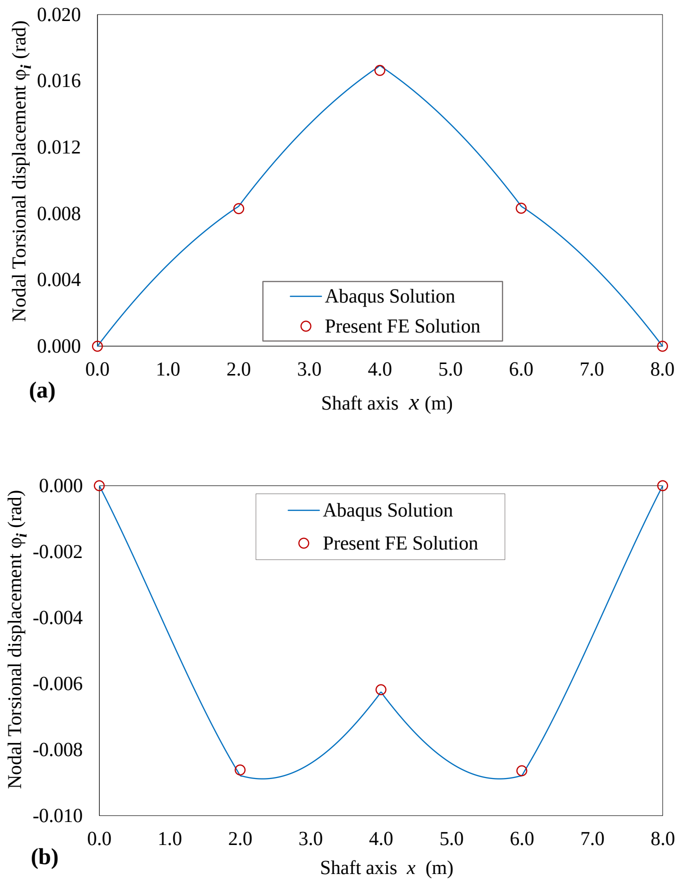

Fig. 8: Nodal torsional displacements for Static and dynamic responses of clamped-clamped shaft under various harmonic torsions

VIII. SUMMARY AND CONCLUSION

From the numerical results conducted throughout this study, the following concluding remarks are made:

- The dynamic equation of motion for torsional vibration and related boundary conditions for shafts under various harmonic twisting moments are derived via Hamilton's principle.
- Exact closed-form solutions of steady state torsional response of shafts are obtained for cantilever and simply-supported shafts.
- The exact closed-form solution derived is successfully used to formulate a family of exact shape functions which based on the homogeneous solution of the governing torsional equation.
- The exact shape functions are used to formulate a super-convergent finite beam element for the shafts. The proposed beam element has a two nodes and two degrees of freedom.
- The present exact closed form solution and finite element formulation developed in this study are able to efficiently capture the quasi-static and steady state response of beams under harmonic torsional loading. It is also capable of extracting the eigen-frequencies and eigen-modes.
- The new beam element involves no discretization errors and generally provides excellent results compared with Abaqus finite element solution while keeping the number of degrees of freedom a minimum.
- Comparison with established Abaqus finite beam element and exact solutions available in the literature demonstrates the validity and accuracy of the present exact closed form solution and finite element formulation.

REFERENCES

[1] C. K. Rao, Torsional frequencies and mode shapes of generally constrained shafts and piping, Journal of Sound and Vibration, 125(1), (1988), 115–121.

[2] E. J. Sapountzakis and V. G. Mokos, Warping shear stresses in non-uniform torsion by BEM, Computational Mechanics, 30(2), (2003), 131–142.

[3] D.V. Mallick and R. Dungar, Dynamic characteristics of core wall structures subjected to torsion and bending. The Structural Engineer, 55, (1977), 251-261.

[4] J. W. Zu and R. P. Han, Natural Frequencies and Normal Modes of a Spinning Timoshenko Beam With General Boundary Conditions, Journal of Applied Mechanics 59(2) , (1992), 197-204.

[5] H. F. Bauer, Vibration of a Rotating Uniform Beam, Part 1: Orientation in the Axis of Rotation, Journal of Sound and Vibration 72, (1981), 177–189.

[6] H. P. Lee, dynamic response of a rotating Timoshenko shaft subject to axial force and moving loads, Journal of Sound and Vibration, 1995, 181 (1), 169-177.

[7] K. Koser and F. Pasin, Torsional vibrations of the drive shafts of mechanisms, Journal of Sound and Vibration, 199, (1997), 559-565.

[8] J. Melanson and J.W. Zu, Free vibration and stability analysis of internally damped rotating shafts with general boundary conditions, Journal of Vibration and Acoustics, 120(3), (1998), 776-783.

[9] S. C. Lin and K. M. Hsiao, Vibration Analysis of A Rotating Timoshenko Beam, Journal of Sound and Vibration 240(2), (2001), 303-322.

[10] S. A. Hosseini and S.E. Khadem, Free vibrations analysis of a rotating shaft with nonlinearities in curvature and inertia, Mechanism and Machine Theory, 44, (2009), 272-288.

[11] T. Yokoyama, Free vibration characteristics of rotating Timoshenko beams, International Journal of Mechanical Sciences, 30(10), (1988), 743-755.

[12] Y. A. Khulief, Vibration frequencies of a rotating tapered beam with end mass, Journal of Sound and Vibration 134, (1989), 87-97.

[13] L. W. Chen and D. M. Ku, Finite element analysis of natural whirl speeds of rotating shafts, Computer and Structures, 40, (1991), 741-747.

[14] M. Y. Chang, J. K. Chen, and C. Y. Chang, A simple spinning laminated composite shaft model, International Journal of Solids and Structures, 41(3-4), (2004), 637–662.