International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 3 Issue 1, January - 2014

Exact Solution for the Flow of Oldroyd-B Fluid Between Coaxial Cylinders

Vatsala Mathur, Kavita Khandelwal
Department of Mathematics
Malaviya National Institute of Technology, Jaipur-302017

Abstract

This paper deals with unsteady unidirectional transient flow of Oldroyd-B fluid between two infinitely long co-axial circular cylinders. At time $t=0^+$, the motion is produced by a constant pressure gradient & the inner cylinder start moving along its axis of symmetry with the constant velocity. The velocity field of the flow of fluid with fractional derivative is obtained by using Hankel and Laplace transforms. The obtained result is presented in terms of the generalized G-functions. Key difference this paper brings from previous work is that in this case inner cylinder is moving with constant velocity. The influence of different values of parameters, constants and fractional coefficients on the velocity field is also analyzed using graphical illustration.

Keywords - *Coaxial cylinders; Fractional calculus; Hankel transform; Laplace transform; Oldroyd-B fluids; Velocity field.*

1. Introduction

Fluids are generally classified based on their rheological properties. The simplest classification is Newtonian fluid. These fluids are represented using Navier-Stokes theory. The fluids which do not obey Newton's law of viscosity $(\tau = \mu \frac{dv}{dy})$, are described as non- Newtonian fluids. Examples of such fluids are blood, saliva, semen, lava, gums, slurries, emulsions, synovial fluid, butter, cheese, jam, ketchup, soup, mayonnaise etc. These fluids have complex molecular structure with non-linear viscoelastic behavior. To study non- Newtonian fluids, many models have been used. Out of these, differential type [1] and rate type [2] have received most of the attention.

Tong [3] used constitutive relation for the flow of non-Newtonian fluid with fractional derivative in an annular pipe as follows

$$(1+\lambda\partial_t)\tau = \mu(1+\lambda_r\partial_t)\partial_r v(r,t), \qquad (1)$$

where τ is tangential tension, μ is the viscosity, v is the velocity, λ and λ_r are relaxation and retardation times respectively.

The starting point of the fractional derivative model of non-Newtonian fluid is usually a classical differential equation being modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operator [4]. Using fractional approach, the constitutive relation of the generalized Oldroyd-B fluid can be written as

$$(1+\lambda D_t^{\alpha})\tau = \mu(1+\lambda_r D_t^{\beta})\partial_r v(r,t), \qquad (2)$$

where D_t^{α} and D_t^{β} are fractional operators and are defined as [5]

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{\alpha}} d\tau, & 0 \le \alpha < 1; \\ \dfrac{d}{dt} f(t), & \alpha = 1, \end{cases} \quad (3)$$

where $\Gamma(.)$ is the Gamma function. When $\alpha = \beta = 1$, eq. (2) simplified as eq. (1).

For non-Newtonian fluids, the first exact solution corresponding to motions of Oldroyd-B fluids in cylindrical domains seem to be those of Waters & King [6]. Fetecau [7] worked on unsteady unidirectional transient flows of an Oldroyd-B fluid in unbounded domains which geometrically are axisymmetric pipe-like. He used the theorem of Steklov to obtain exact solutions for flows satisfying no-slip boundary conditions. The unsteady rotational flow of a generalized second grade fluid through a circular cylinder has been considered by Kamran [8]. Exact solutions for the velocity field and the shear stress corresponding to the unsteady flows of a generalized Oldroyd-B fluid in an infinite circular cylinder subject to a longitudinal time dependent shear stress have been obtained by Rubbab [9]. M. Kamran [10] concluded unsteady linearly accelerating flow of a fractional second grade fluid through a circular cylinder. Various other studies have been done recently on non-Newtonian fluids [11-17].

The aim of this paper is to study the flow of Oldroyd-B fluid with fractional derivative between two coaxial cylinders. The solution is obtained by using Hankel and Laplace transform. At time $t = 0^+$ a constant pressure gradient applied and the inner cylinder moves with constant velocity & the outer cylinder held fixed. The obtained result is presented in terms of the generalized G functions.

2. Governing equations

We consider the unsteady flow of an incompressible Oldroyd-B fluid in coaxial cylinders. Further, following assumptions are considered during this mathematical study. The fluid velocity at the direction of the pipe radius is assumed to be zero. The flows are assumed to be axisymmetric. The axial velocity is assumed to be only relevant to the cylinder radius.
The equation of axial flow motion is

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \tau}{\partial r} + \frac{1}{r}\tau - \frac{\partial p}{\partial z}, \quad (4)$$

where ρ is the constant density of the fluid.

Putting eq. (2) in eq. (4), we get

$$\left(1+\lambda D_t^{\alpha}\right)\frac{\partial v}{\partial t} = A + \lambda A \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \upsilon\left(1+\lambda_r D_t^{\beta}\right)\left(\partial_r^2 + \frac{1}{r}\partial_r\right) v(r,t), \quad (5)$$

where $\upsilon = \dfrac{\mu}{\rho}$ is the kinematical viscosity and $-A\rho = \dfrac{\partial p}{\partial z}$ is the constant pressure gradient that acts on the liquid in the z-direction.

3. Flow through the annular region

Let us consider an incompressible Oldroyd-B fluid in infinite coaxial circular cylinders. At time $t = 0$, fluid is assumed to be stationary. At time $t = 0^+$, a constant pressure gradient applied and the inner cylinder moves with constant velocity and the outer cylinder held fixed. Consider that the radius of inner and outer cylinders are R_1 and $R_2 (> R_1)$ respectively.

The initial and boundary conditions are

$$v(r,0) = 0, \quad \partial_t v(r,0) = 0, \quad R_1 \le r \le R_2, \quad (6)$$

$$v(R_1,t) = f, \quad v(R_2,t) = 0, \quad t > 0, \quad (7)$$

where f is constant.

Making the change of unknown function

$$v(r,t) = V(r) + u(r,t), \quad (8)$$

where

$$V(r) = \frac{A}{4\upsilon}(R_2^2 - r^2) + \frac{A}{4\upsilon}\frac{(R_2^2 - R_1^2)}{\ln(R_2/R_1)}\ln(r/R_2). \quad (9)$$

Putting eq. (8) in eq. (5), we obtain

$$(1+\lambda D_t^{\alpha})\frac{\partial u(r,t)}{\partial t}=\upsilon(1+\lambda_r D_t^{\beta})\left(\partial_r^2+\frac{1}{r}\partial_r\right)u(r,t)+\lambda A\frac{t^{-\alpha}}{\Gamma(1-\alpha)}-\lambda_r A\frac{t^{-\beta}}{\Gamma(1-\beta)}. \quad (10)$$

Putting eq. (8) in eqs. (6) & (7), we obtain

$$u(r,0)=-V(r),\qquad \partial_t u(r,0)=0, \quad (11)$$

$$u(R_1,t)=f,\qquad u(R_2,t)=0. \quad (12)$$

The Hankel Transform method with respect to *r* is used and is defined as follows

$$\bar{u}=\int_{R_1}^{R_2} r u(r,s)\phi_1(s_n,r)dr. \quad (13)$$

The inverse Hankel Transform is

$$u(r,s)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 J_0^2(R_2 s_n)\bar{u}(s_n,s)\phi_1(s_n,r)}{J_0^2(R_1 s_n)-J_0^2(R_2 s_n)}, \quad (14)$$

where

$\phi_1(s_n,r)=Y_0(R_1 s_n)J_0(s_n r)-J_0(R_1 s_n)Y_0(s_n r)$, s_n is the positive root of $\phi_1(s_n,R_2)=0$.

Applying the Hankel transform in eq. (10), we obtain

$$(1+\lambda D_t^{\alpha})\frac{\partial \bar{u}(s_n,t)}{\partial t}=-\upsilon s_n^2(1+\lambda_r D_t^{\beta})\bar{u}(s_n,t)-\frac{2\upsilon}{\pi}f-\frac{2\upsilon\lambda_r f}{\pi}\frac{1}{\Gamma(1-\beta)}t^{-\beta}+\frac{2\lambda A g(s_n)t^{-\alpha}}{\pi s_n^2\Gamma(1-\alpha)}-\frac{2\lambda_r A g(s_n)t^{-\beta}}{\pi s_n^2\Gamma(1-\beta)}, \quad (15)$$

where

$$g(s_n)=\left[\frac{R_2\pi s_n}{2}\{Y_0(R_1 s_n)J_1(R_2 s_n)-J_0(R_1 s_n)Y_1(R_2 s_n)\}-1\right].$$

Applying the Hankel transform of eq. (11), we obtain

$$\bar{u}(s_n,0)=-\frac{2A}{\pi s_n^4\upsilon}\left[\frac{J_0(R_1 s_n)}{J_0(R_2 s_n)}-1\right],\ \partial_t\bar{u}(s_n,0)=0. \quad (16)$$

Applying Laplace transform of eq. (15) and using eq. (16), we obtain

$$\bar{\bar{u}}(s_n,s)=\left[-\frac{2A}{\pi s_n^2\upsilon}\left\{\frac{J_0(R_1 s_n)}{J_0(R_2 s_n)}-1\right\}\right]\frac{(1+\lambda s^{\alpha}+\upsilon s_n^2\lambda_r s^{\beta-1})}{s_n^2(s+\lambda s^{\alpha+1}+\upsilon s_n^2+\upsilon s_n^2\lambda_r s^{\beta})}-\frac{2\upsilon f}{\pi}\frac{1}{s(s+\lambda s^{\alpha+1}+\upsilon s_n^2+\upsilon s_n^2\lambda_r s^{\beta})}-\frac{2\upsilon\lambda_r f}{\pi}\frac{1}{s^{1-\beta}(s+\lambda s^{\alpha+1}+\upsilon s_n^2+\upsilon s_n^2\lambda_r s^{\beta})}+\frac{2\lambda A g(s_n)}{\pi s_n^2}\frac{1}{s^{1-\alpha}(s+\lambda s^{\alpha+1}+\upsilon s_n^2+\upsilon s_n^2\lambda_r s^{\beta})}-\frac{2\lambda_r A g(s_n)}{\pi s_n^2}\frac{1}{s^{1-\beta}(s+\lambda s^{\alpha+1}+\upsilon s_n^2+\upsilon s_n^2\lambda_r s^{\beta})}. \quad (17)$$

Applying Inverse-Laplace transform of eq. (17) and taking into account the following result [18]

$$G_{a,b,c}(d,t)=L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=\sum_{j=0}^{\infty}\frac{\mathrm{d}^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]};$$

$$\mathrm{Re}(ac-b)>0,\ \left|\frac{\mathrm{d}}{\mathrm{q}^{\mathrm{a}}}\right|<1. \quad (18)$$

We obtain

$$\bar{u}(s_n,t)=\left[-\frac{2A}{\pi s_n^2\upsilon}\left\{\frac{J_0(R_1 s_n)}{J_0(R_2 s_n)}-1\right\}\right]\left\{\frac{1}{s_n^2}-\frac{1}{s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^k G_{\alpha,\beta k-m-2,m+1}(-\lambda^{-1},t)\right\}$$
$$-\frac{2}{\pi}f\frac{1}{s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^k G_{\alpha,\beta k-m-2,m+1}(-\lambda^{-1},t)$$
$$-\frac{2}{\pi}\lambda_r f\frac{1}{s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^k G_{\alpha,\beta k-m+\beta-2,m+1}(-\lambda^{-1},t)$$
$$+\frac{2\lambda A g(s_n)}{\pi s_n^2}\frac{1}{\upsilon s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^k G_{\alpha,\beta k-m+\alpha-2,m+1}(-\lambda^{-1},t)$$
$$-\frac{2\lambda_r A g(s_n)}{\pi s_n^2}\frac{1}{\upsilon s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^k G_{\alpha,\beta k-m+\beta-2,m+1}(-\lambda^{-1},t)\ . \quad (19)$$

The expression of the velocity field can be written as

$$v(r,t)=V(r)+\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_2s_n)\phi_1(s_n,r)}{\left[J_0^2(R_1s_n)-J_0^2(R_2s_n)\right]}$$
$$\left[\left\{-\frac{A}{s_n^2\upsilon}\left(\frac{J_0(R_1s_n)}{J_0(R_2s_n)}-1\right)\right\}\right.$$
$$\times\left\{1-\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^kG_{\alpha,\beta k-m-2,m+1}(-\lambda^{-1},t)\right\}$$
$$-f\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^kG_{\alpha,\beta k-m-2,m+1}(-\lambda^{-1},t)$$
$$-\lambda_r f\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^kG_{\alpha,\beta k-m+\beta-2,m+1}(-\lambda^{-1},t)$$
$$+\frac{\lambda Ag(s_n)}{\upsilon s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^kG_{\alpha,\beta k-m+\alpha-2,m+1}(-\lambda^{-1},t)$$
$$\left.-\frac{\lambda_r Ag(s_n)}{\upsilon s_n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\upsilon s_n^2}{\lambda}\right)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\lambda_r^kG_{\alpha,\beta k-m+\beta-2,m+1}(-\lambda^{-1},t)\right].\quad(20)$$

4. Results

As shown in below diagrams, the velocity $v(r,t)$ given by eq. (20) has been drawn against r for different values of the time t, f and some other relevant parameters.

Figure 1 is showing the time dependency on the fluid motion. It can also be clearly seen that the velocity increases, when time increases.

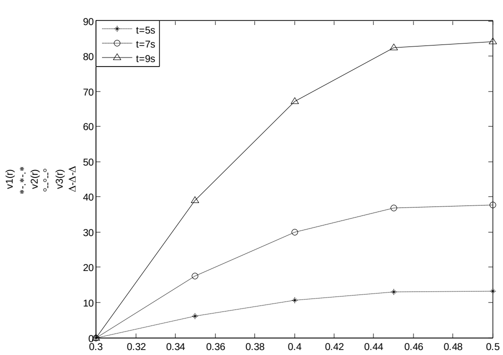

Fig. 1: Profiles of the velocity *v(r,t)* given by eq. (20) for R_1=0.3, R_2=0.5, f=-3, v=0.035, λ=12, λ_r=2.2, α=0.9, β=0.6, A=4 and different values of t.

Figure 2 is showing the dependency of the kinematic viscosity υ on the fluid motion. It can also be clearly seen that the velocity decreases, when kinematic viscosity υ increases.

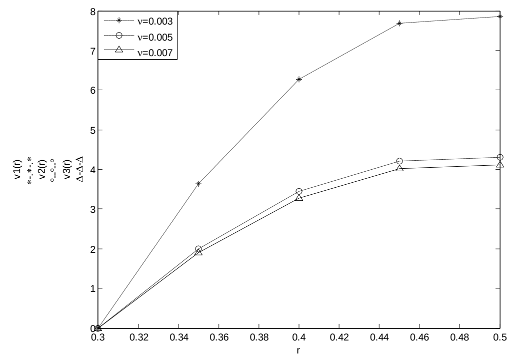

Fig. 2: Profiles of the velocity *v(r,t)* given by eq. (20) for *R_1=0.3, R_2=0.5, f=-3, t=6s, λ=9, λ_r=4, α=0.3,* β=*0.3, A=4* and different values of *v*.

Figure 3 is showing the dependency of the relaxation time λ on the fluid motion. It can also be clearly seen that the velocity decreases, when λ increases.

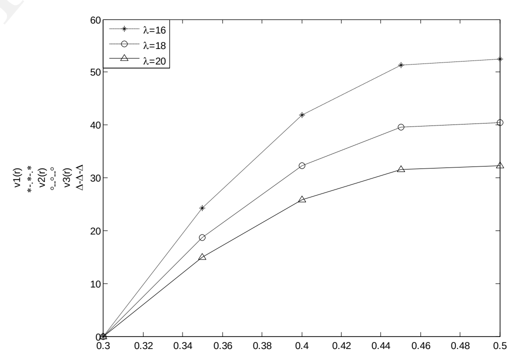

Fig. 3: Profiles of the velocity *v(r,t)* given by eq. (20) for *R_1=0.3, R_2=0.5, f=-3, t=5s,v=0.04, λ_r=7,α=0.3,* β=*0.3, A=4* and different values of *λ*.

Figure 4 is showing the dependency of the retardation time λ_r on the fluid motion. It can also be clearly seen that the velocity increases, when λ_r increases.

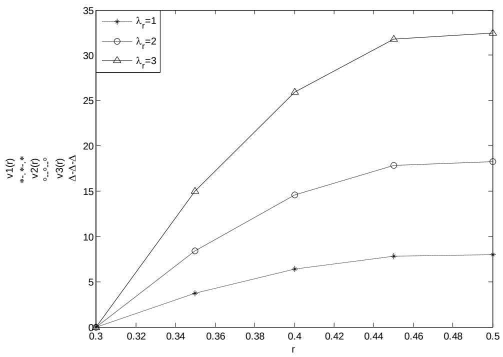

Fig. 4: Profiles of the velocity *v(r,t)* given by eq. (20) for *R_1=0.3, R_2=0.5, f=-3,t=5s, v=0.04, λ=8, α=0.3,* β*=0.9, A=4* and different values of *λ_r*.

Figure 5 is showing the dependency of the fractional parameter α on the fluid motion. It can also be clearly seen that the velocity increases, when α increases.

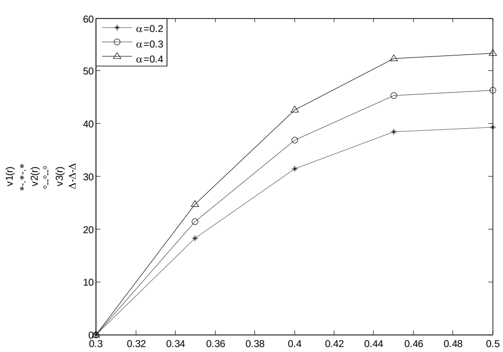

Fig. 5: Profiles of the velocity *v(r,t)* given by eq. (20) for *R_1=0.3, R_2=0.5, f=-3, t=6s, v=0.045, λ=25, λ_r=8,* β*=0.5, A=4* and different values of *α*.

Figure 6 is showing the dependency of the fractional parameter β on the fluid motion. It can also be clearly seen that the velocity decreases, when β increases.

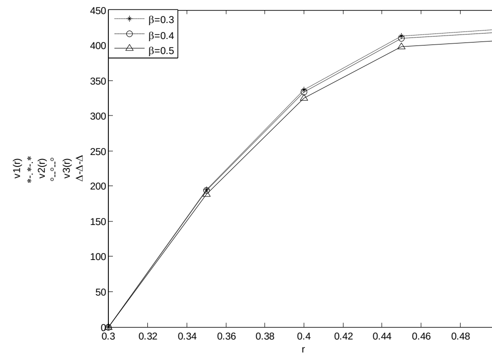

Fig. 6: Profiles of the velocity *v(r,t)* given by eq. (20) for *R_1=0.3, R_2=0.5, f=-3, t=6s,v=0.04, λ=8, λ_r=5.5, α=1, A=4* and different values of β.

Figure 7 is showing the dependency of *f* on the fluid motion. It can also be clearly seen that the velocity decreases, when *f* increases.

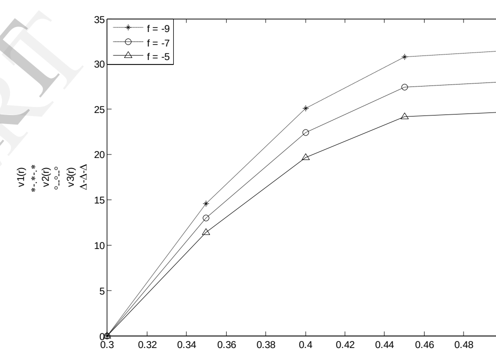

Fig. 7: Profiles of the velocity *v(r,t)* given by eq. (20) for *R_1=0.3, R_2=0.5, t=5s, v=0.045, λ=14, λ_r=2.8, α=0.8,* β=0.5, *A=4* and different values of *f*.

Figure 8 is showing the dependency of A on the fluid motion. It can also be clearly seen that the velocity increases, when A increases.

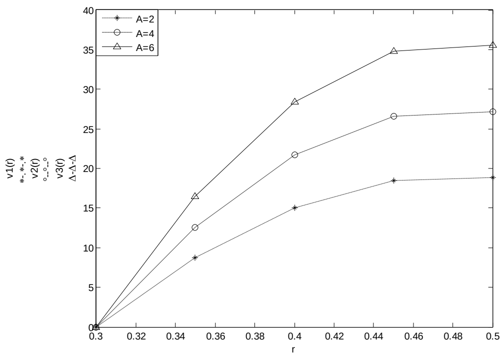

Fig. 8: Profiles of the velocity *v(r,t)* given by eq. (20) for $R_1=0.3$, $R_2=0.5$, $f=-3$, $t=5s, v=0.04$, $\lambda=11, \lambda_r=2.5, \alpha=0.9$, β=0.6 and different values of *A*.

In all of above, the root s_n has been approximated by $\dfrac{(2n-1)\pi}{2(R_2-R_1)}$.

5. Conclusions

The main objective of this paper is to provide exact solution for the velocity field for Oldroyd-B fluid between two coaxial circular cylinders where inner cylinder is moving with constant velocity & the outer cylinder is fixed. This solution is obtained by using Hankel transform and Laplace transform methods and the result is presented in terms of generalized G functions. This solution satisfies the governing equation and all imposed initial and boundary conditions. The velocity field is also analyzed using graphical illustration for various parameters, constants and fractional coefficients.

References

[1] Dunn, J.E.; Rajagopal, K.R., (1995). Fluids of differential type: critical review and thermodynamic analysis. Int. J. Eng. Sci., 33, 689-729.
[2] Rajagopal, K.R.; Srinivasa, A.R., (2000). A thermodynamical frame-work for rate type fluid models. J. Non-Newtonian Fluid Mech., 88, 207-227.
[3] Tong, D.; Wang, R.; Yang, H., (2005). Exact solutions for the flow of non-Newtonian fluid with fractional derivative in an annular pipe. Science in China Ser. G Physics, Mechanics & Astronomy, 48, 485-495.
[4] Qi, H.; Jin, H., (2006). Unsteady rotating flows of a viscoelastic fluid with the fractional Maxwell model between coaxial cylinders. Acta Mechanica Sinica, 22, 301–305.
[5] Podlubny, I., (1999). Fractional differential equations. Academic Press, San Diego.
[6] Waters, N.D.; King M.J., (1971). The unsteady flow of an Elastico-viscous liquid in a straight pipe of circular cross section. J. Phys. D Appl. Phys., 4, 204–211.
[7] Fetecau, C., (2004). Analytical solutions for non-Newtonian fluid flows in pipe-like domains. Int. J. Non-Linear Mechanics, 39, 225 – 231.
[8] Kamran, M.; Imran, M.; Athar, M., (2010). Exact solutions for the unsteady rotational flow of a generalized second grade fluid through a circular cylinder. Nonlinear Analysis: Modelling and Control, 15, 437–444.
[9] Rubbab, Q., Husnine, S.M., Mahmood, A., (2009). Exact solutions of generalized Oldroyd-B fluid subject to a time-dependent shear stress in a pipe. Journal of Prime Research in Mathematics, 5, 139-148.
[10] Kamran, M.; Athar, M.; Imran, M., (2011). On the unsteady linearly accelerating flow of a fractional second grade fluid through a circular cylinder. International Journal of Nonlinear Science, 11, 317-324.
[11] Amir, M.; Fetecau, C.; Imran, S., (2008). Exact solutions for some unsteady flows of generalized second grade fluids in cylindrical domains. Journal of Prime Research in Mathematics, 4, 171-180.
[12] Kamran, M.; Imran, M.; Athar, M.; Imran, M.A., (2012). On the unsteady rotational flow of fractional Oldroyd-B fluid in cylindrical domains. Meccanica, 47, 573–584.
[13] Athar, M.; Awan, A.U.; Fetecau, C., (2012). A note on the unsteady flow of a fractional Maxwell fluid through a circular cylinder. Acta Mech. Sin., 28(2), 308–314.
[14] Jamil, M.; Fetecau, C.; Fetecau, C., (2012). Unsteady flow of viscoelastic fluid between two cylinders using fractional Maxwell model. Acta Mech. Sin., 28(2), 274–280.
[15] Athar, M.; Kamran, M.; Fetecau, C., (2010). Taylor–Couette flow of a generalized second grade fluid due to a constant couple. Nonlinear Analysis: Modelling and Control, 15(1), 3–13.
[16] Imran, M.; Kamran, M.; Athar, M.; Zafar, A.A., (2011). Taylor–Couette flow of a fractional second grade fluid in an annulus due to a time-dependent couple. Nonlinear Analysis: Modelling and Control, 16(1), 47–58.
[17] Athar, M.; Fetecau, C.; Kamran, M.; Sohail, A.; Imran, M., (2011). Exact solutions for unsteady axial Couette flow of a fractional Maxwell fluid due to an accelerated shear. Nonlinear Analysis: Modelling and Control, 16(2), 135–151.
[18] Lorenzo, C.F.; Hartley, T.T., (1999). Generalized functions for the fractional calculus. NASA/TP-1999-209424/REV1