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Extension of A Drℓ - Group

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Abstract: In this paper we introduce the idea of a "DRℓ - group is a direct product of a Brouwerian Algebra and a commutative ℓ - group".

Key words: commutative ℓ - group, Brouwerian Algebra, DRℓ - group.

1. Preliminaries

Definition 1.1 [4]

A non – empty set G is called a commutative ℓ – group if and only if

(i) (G, +) is an abelian group

(ii) (G, ≤) is a lattice

(iii) If $x \leq y$, then $a + x + b \leq a + y + b$, for all a, b, x, y in G.

(or)

$(a + x + b) \vee (a + y + b) = (a + x \vee y + b)$

$(a + x + b) \wedge (a + y + b) = (a + x \wedge y + b)$, *for all a, b, x, y in G.*

Definition 1.2 [1], [4]

A non – empty set B is called a Brouwerian Algebra if and only if

(i) (B, ≤) is a lattice

(ii) B has a least element

(iii) To each a, b in B, there is a least $x = a – b$ in B such that $b \vee x \geq a$

Definition 1.3 [4], [5]

A lattice L is called a residuated lattice if

(i) (L, .) is an ℓ– group.

(ii) Given a, b in L, there exist the largest x, y such that

$bx \leq a$ *and* $yb \leq a$.

Definition 1.3 [4]

A system A = (A, +, ≤) is called dually residuated lattice ordered group (simply DRℓ– group) if and only if

(i) (A, +) is an abelian group.

(ii) (A, ≤) is a lattice.

(iii) $b \leq c \Rightarrow a + b \leq a + c$, *for all a, b, c in A*

(iv) Given a, b in A, there exist a least element $x = a - b$ *in A such that* $b + x \geq a$.

Definition 1.4 [4]

A system A = (A, +, ∨, ∧) is called a DRℓ– group if and only if

(i) (A, +) is an abelian group.

(ii) (A, ∨, ∧,) is a lattice.

(iii) $a + (b \vee c) = (a + b) \vee (a + c)$,

$a + (b \wedge c) = (a + b) \wedge (a + c)$, *for all a, b, c in A.*

(iv) $x + (y - x) \geq y$,

$x - y \leq (x \vee z) - y$,

$(x + y) - y \leq x$, *for all x, y, z in A.*

Remark [4]

Two definitions for DRℓ– group are equivalent.

Examples 1.1 [4]

Commutative ℓ - group, Brouwerian Algebra and Boolean ring are DRℓ - groups.

Extension of a DRℓ - group

Theorem : 1.1

Any DRℓ - group A is the direct product of a Brouwerian Algebra B and a commutative ℓ - group G if and only if

i) $(a + b) - (c + c) \geq (a - c) + (b - c)$ *and*

ii) $(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$,

for all a, b, c in A and any pair of positive integers m, n.

Proof :

Assume that

$$(a + b) - (c + c) \geq (a - c) + (b - c) \qquad \rightarrow (1)$$

$$(ma + nb) - (a + b) \geq (ma - a) + (nb - b) \qquad \rightarrow (2)$$

To prove A = B x G

Let a, b, c in A be arbitrary

$\Rightarrow (a + b) - c \leq (a - c) + b$, by property 11 [4]

$\Rightarrow [(a + b) - c] - c \leq [(a - c) + b] - c$

$= (a - c) + (b - c)$

$$\Rightarrow (a + b) - (c - c) \leq (a - c) + (b - c) \qquad \rightarrow (3)$$

From (1) and (3), we get

$$(a + b) - (c + c) = (a - c) + (b - c) \qquad \rightarrow (4)$$

Also, $(ma + nb) - a \leq (ma - a) + nb$, by property 11 [4]

$\Rightarrow [(ma + nb) - a] - b \leq [(ma - a) + nb] - b$

$$\Rightarrow (ma + nb) - (a + b) \leq (ma - a) + (nb - b) \qquad \rightarrow (5)$$

Form (2) and (5), we have

$$(ma + nb) - (a + b) = (ma - a) + (nb - b) \quad \rightarrow (6)$$

Let $B = \{a \,/\, a + a - a = 0\}$

$G = \{a \,/\, a + a - a = a\}$

Claim (1): B is a Brouwerian Algebra

(i) Closed with respect to $\vee$ and $\wedge$

Let a in B be arbitrary

$$\Rightarrow (a + a) - a = 0 \leq 0$$

$$\Rightarrow (a + a) - a \leq 0$$

$$\Rightarrow [(a + a) - a] + a \leq 0 + a$$

$$\Rightarrow a + a \leq a$$

Now $0 = (a + a) - a \leq (a - a) + a$, by property 11 [4]

$$\Rightarrow 0 \leq a$$

$$\Rightarrow 0 + a \leq a + a$$

$$\Rightarrow a \leq a + a$$

Thus $a + a = a$ $\rightarrow (7)$

Hence B is closed under " +"

Let a, b in B be arbitrary

Then $(a - b) + (a - b) = (a + a) - (b + b)$, by (4)

$= a - b$, by (7)

$\Rightarrow a - b$ in B

$\Rightarrow (a - b) + b$ in B

$\Rightarrow a \vee b$ in B, by property 7 [4]

Also, $(a + b) - (a \vee b) = (a + b) - [(a \vee b) + (a \vee b)]$, since $a \vee b$ in B

$= [\, a - (a \vee b)] + [\, b - (a \vee b)],$ by (4)

$= [(a - a) \wedge (a - b)] + [(b - a) \wedge (b - b)],$ by property 4 [4]

$= [\, 0 \wedge (a - b)] + [(b - a) \wedge 0]$

$= 0 + 0$

$\Rightarrow (a + b) - (a \vee b) = 0$

$\Rightarrow a + b = a \vee b$

Let a, b in B $\Rightarrow a + b,\ a \vee b$ in B

$\Rightarrow (a + b) - (a \vee b)$ in B

$\Rightarrow a \wedge b$ in B

(ii) (B, $\vee$, $\wedge$) is a lattice

Idempotent law

Let a in B be arbitrary

Then $a \vee a = a + a$

$= a$

$a \wedge a = (a + a) - (a \vee a)$

$= a + a - a$

$= a$

Thus $a \vee a = a$; $a \wedge a = a$, for all a in B.

Commutative law:

Let a, b in B be arbitrary

Then $a \vee b = a + b$

$= b + a$

$= b \vee a$

$a \wedge b = (a + b) - (a \vee b)$, by property 8 [4]

$= (b + a) - (b \vee a)$

$= b \wedge a$

Thus $a \vee b = b \vee a$; $a \wedge b = b \wedge a$, for all a, b, in B

Associative Law :

Let a, b, c in B be arbitrary.

Then $a \vee (b \vee c) = a + (b + c)$

$= (a + b) + c$

$= (a \vee b) \vee c$

$a \wedge (b \wedge c) = [a + (b \wedge c)] - [a \vee (b \wedge c)]$, by property 8 [4]

$= [a + (b \wedge c)] - [a + (b \wedge c)]$

$= 0$

$(a \wedge b) \wedge c = [(a \wedge b) + c] - [(a \wedge b) \vee c]$

$= [(a \wedge b) + c] - [(a \wedge b) + c]$

$= 0$

Thus $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all a, b, c in B

Absorption law :

Let a, b in B be arbitrary

Then $a \vee (a \wedge b) = a + (a \wedge b)$

$= a + [(a + b) - (a \vee b)]$

$= a + [(a + b) - (a + b)]$

$= a$

$a \wedge (a \vee b) = [a + (a \vee b)] - [a \vee (a \vee b)]$

$= [a + (a \vee b)] - [(a \vee a) \vee b]$

$= a + (a \vee b) - (a \vee b)$

$= a$

Thus $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$, for all a, b in B

Hence (B, $\vee$, $\wedge$) is a lattice.

(iii) B has a least element :

Let a in B be arbitrary

Then $0 = (a + a) - a \leq (a - a) + a$

$\Rightarrow 0 \leq a$, for all a in B

Hence B has a least element.

(iv) To each a, b in B , there exist a least element x = a –b in B such that $b \vee x \geq a$:

Let a, b in B be arbitrary

$\Rightarrow$ there exist a least element x = a – b in B

Now $b \vee x = b + x$

$= b + (a - b)$

$= a \geq a$

Thus to each a, b in B, there exist a least element x = a –b in B such that $b \vee x \geq a$

Hence B is a Brouwerian Algebra

Claim (2) : G is a commutative ℓ - group

(i) (G, +) is an abelian group

Closure law :

Let a, b in G be arbitrary

Then $[(a + b) + (a + b)] - (a + b) = (2a + 2b) - (a + b)$

$= (2a - a) + (2b - b)$, by (6)

$= a + b$

$\Rightarrow a + b$ in G

Thus a, b in G $\Rightarrow$ a + b in G

Clearly, + is both associative and commutative in G, since G is a subset of A.

Existence of Identity:

Let a in G be arbitrary.

Clearly 0 in G, since 0 = 0 + 0 – 0

Then a + 0 = 0 + a = a , for all a in G.

Existence of Inverse :

Let a in G be arbitrary

Then (-a) + (-a) – (-a) = – a – a + a

= -a

$\Rightarrow$ -a in G

Now, a + (-a) = (-a) + a = 0

Hence (G, +) is an abelian group

(ii) (G $\vee$, $\wedge$) is a lattice :

Let a, b in G be arbitrary.

$\Rightarrow$ a, -b, b in G

$\Rightarrow$ a –b, b in G

$\Rightarrow$ (a – b) + b in G

$\Rightarrow$ a $\vee$ b in G, by property 7 [4]

Also a, b in G $\Rightarrow$ a + b , a $\vee$ b in G

$\Rightarrow$ (a + b) – (a $\vee$ b) in G

$\Rightarrow$ a $\wedge$ b in G

Idempotent law:

Let a in G be arbitrary

Then $a \vee a = (a - a) + a$

$= a$

$a \wedge a = (a + a) - (a \vee a)$

$= (a + a) - a$

$= a$

Thus $a \vee a = a$; $a \wedge a = a$, for all a in G

Commutative law:

Let a, b in G be arbitrary

Then $a \vee b = (a + b) - (a \wedge b)$

$= [(a + b) - a] \vee [(a + b) - b]$

$= b \vee a$

$a \wedge b = (a + b) - (a \vee b)$

$= (b + a) - (b \vee a)$

$= b \wedge a$

Thus $a \vee b = b \vee a$; $a \wedge b = b \wedge a$, for all a, b in G

Associative law :

Let a, b, c in G be arbitrary

Then $a \vee (b \vee c) = [a - (b \vee c)] + (b \vee c)$

$= a$

$(a \vee b) \vee c = [(a \vee b) - c] + c$

$= a \vee b$

$= (a - b) + b$

$= a$

Therefore, $a \vee (b \vee c) = (a \vee b) \vee c$, for all a, b, c in G.

Also, $a \wedge (b \wedge c) = [\, a + (b \wedge c)] - [\, a \vee (b \wedge c)]$

$= [\, a + (b \wedge c) - ([a - (b \wedge c)] + (b \wedge c))$

$= a + (b \wedge c) - a$

$= b \wedge c$

$= (b + c) - (b \vee c)$

$= (b + c) - [(b - c) + c\,]$

$= b + c - b$

$= c$

$(a \wedge b) \wedge c = [(a \wedge b) + c] - [(a \wedge b) \vee c]$

$= [(a \wedge b) + c] - ([(a \wedge b) - c] + c)$

$= (a \wedge b) + c - (a \wedge b) = c$

Therefore, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all a, b, c in G.

Absorption Law :

Let a, b in G be arbitrary

Then $a \vee (a \wedge b) = [\, a - (a \wedge b)] + (a \wedge b)$

$= a$

$a \wedge (a \vee b) = [\, a - (a \vee b)] - [a \vee (a \vee b)]$

$= [a + (a \vee b)] - [(a \vee a) \vee b]$

$= a + (a \vee b) - (a \vee b) = a$

Thus $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$, for all a, b in G

Therefore, $(G, \vee, \wedge)$ is a lattice

(iii) $a + (b \vee c) = (a + b) \vee (a + c)$,
$a + (b \wedge c) = (a + b) \wedge (a + c)$, for all a, b , c in G :

Let a, b, c in G be arbitrary

Then $a + (b \vee c) = a + [(b - c) + c]$

$= a + b$

$(a + b) \vee (a + c) = [(a + b) - (a + c)] + (a + c)$

$= a + b$

Therefore, $a + (b \vee c) = (a + b) \vee (a + c)$, for all a, b, c in G

Also, $a + (b \wedge c) = a + [(b + c) - (b \vee c)]$

$= a + ((b + c) – [(b- c) + c])$

$= a + [(b + c) – b]$

$= a + c$

$(a + b) \wedge (a + c) = [(a + b) + (a + c)] – [(a + b) \vee (a + c)]$, by property 8 [4]

$= (a + b) + (a + c) – (a + b)$, by previous result

$= (a + c)$

Therefore, $a + (b \wedge c) = (a + b) \wedge (a + c)$, for all a, b, c in G

Hence G is a commutative ℓ - group.

Claim (3): A = B x G

For any a in A, $y = (a + a) – a$, $x = a - [(a + a) – a]$ implies y in G, x in B

Now $(y + y) – y = [(2a - a) + (2a - a)] – (2a – a)$

$= [(2a + 2a) – (a + a)] – (2a – a)$, by (6)

$= (4a – 2a) – (2a – a)$

$\geq 4a – 2a – a$, since $2a – a \leq a$

$= a$

$\Rightarrow (y + y) – y \geq y$

Also, $(y + y) – y \leq (y – y) + y$, by property 11 [4]

$= y$

$\Rightarrow (y + y) – y \leq y$

Therefore , $(y + y) – y \leq y$

$\Rightarrow$ y in G

y = (a + a) – a

$\Rightarrow \quad y \leq a$

$\Rightarrow \quad x \geq 0$

$\Rightarrow x + x \geq 0 + x$

$\Rightarrow x + x \geq x$

Now, $(a - y) + (a - y) = (a + a) - (y + y)$, by (4)

$= 2a - 2y$

$= 2a - 2\,(2a - a)$

$\Rightarrow \quad x + x = 2a - (4a - 2a)$

We have

$(4a - 2a) + [a - (2a - a)] = (2a - a) + (2a - a) + [a - (2a - a)]$

$\geq (2a - a) + a$

$= 2a$

$\Rightarrow \quad (4a - 2a) + [a - (2a - a)] \geq 2a$

$\Rightarrow \quad 2a - (4a - 2a) \leq a - (2a - a) = x$

$\Rightarrow \quad x + x \leq x$

$\Rightarrow \quad x + x = x$

$\Rightarrow$ x in B

Thus if a in A, then a = x + y , where x in B, y in G

Now, let a = $x' + y'$, where x' in B, y' in G

Then $a + a = (x' + y') + (x' + y')$

$= (x' + x') + (y' + y')$

$= x' + 2y'$, since x' in B

$= (x' + y') + y'$

$\Rightarrow \quad a + a \;=\; a + y'$

$\Rightarrow (a + a) - a \;=\; (a + y') - a$

$\Rightarrow (a + y') - a$ in G

$[(a + y') - a] - y' \;=\; (a + y') - (a + y')$

$= 0$

$\Rightarrow \quad a + y' \;=\; a + y'$

$\Rightarrow a + y' - a \;=\; y'$

$\Rightarrow (a + a) - a \;=\; y'$

Now, $\quad a \;=\; x' + y'$

$\Rightarrow \quad a - y' \;=\; x' \leq x'$

$\Rightarrow \quad a - y' \leq x'$

Also, $\quad x' - (a - y') \;\leq\; (x' - a) + y'$

$= [x' - (x' + y')] + y'$

$= (0 - y') + y' = 0$

$\Rightarrow x' - (a - y') \leq 0$

$\Rightarrow \quad x' \leq a - y'$

Hence $\quad x' = a - y'$

Hence follows that A is the direct product of a Brouwerian Algebra B and a commutative ℓ - group G.

Conversely, assume that A = B x G, where B is a Brouwerian Algebra and G is a commutative ℓ – group.

To prove

(i) $\quad (a + b) - (c + c) \;\geq\; (a - c) + (b - c),$

(ii) (ma + nb) – (a + b) $\geq$ (ma – a) + (nb – b),

for all a, b, c in A and any pair of positive integers m, n.

Let a, b, c in A be arbitrary

$\Rightarrow$ (i). To each [(a – c) + (b – c)], (a + b) in B, there exist a least – (c + c) in B such that (a + b) – (c + c) $\geq$ (a – c) + (b – c),

(ii). To each [(ma – a) + (nb – b)], (ma + nb) in B, there exist a least element – (a + b) in B such that (ma +nb) – (a + b) $\geq$ (ma – a) + (nb – b),

since B is a Brouwerian Algebra

$\Rightarrow$ (i) (a + b) – (c + c) $\geq$ (a –c) + (b – c),

(ii) (ma + nb) – (a + b) $\geq$ (ma – a) + (nb – b),

for all a, b, c in A and any pair of positive integers m, n.

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