International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 2 Issue 3, March - 2013

Fixed Point Theorems For Continuous Mappings Based On Commutativity In A Complete 2 -Metric Space

A. Sangeetha

Assistant Professor, Department of Mathematics, EBET Group of Institutions

Nathakadaiyur, Tirupur D.T.

ABSTRACT

The notion of 2-metric space was first introduced by S.Gahler in 1963. Gahler gave the definition of 2-metric space. After that Ehret (1969), Iseki (1975), Diminie and White (1976), Singh (1979), Rhoades (1979), SharmaAshok K(1980), M.S.Khan and Fisher (1982) S V R Naidu (1986), Ganguly and Chandel (1987) and number of other Mathematicians have worked in this field. The aim of this paper is to study about the fixed point theorems for continuous self - maps satisfying the commuting property in a complete 2-metric space.

INTRODUCTION

In mathematics, a metric space is a set where a notion of distance (called a metric) between elements of the set is defined. The metric space which most closely corresponds to our intuitive understanding of space is the 3-dimensional Euclidean space. In fact, the notion of "metric" is a generalization of the Euclidean metric arising from the four long-known properties of the Euclidean distance. The Euclidean metric defines the distance between two points as the length of the straight line segment connecting them. The geometric properties of the space depend on the metric chosen, and by using a different metric we can construct interesting non-Euclidean geometries such as those used in the theory of general relativity. The 2-metric space was shown to have a unique nonlinear structure, quite different from a metric space. As in other spaces, the fixed point theory of operators has been developed in this space also. A 2-metric is a real function of triple points which abstracts the properties of the area function for Euclidean triangles.

In this paper, we have some useful definitions and two fixed point theorems in complete 2-metric space using the concept of continuous self–mapping and commutative mapping. Throughout this paper,

(i) **(X,d)** is a complete 2-metric space ,

(ii) ϕ is a non-negative real function mapping from R^+ to R^+ such that ϕ is a non-decreasing function, $\phi(t) < t$ for any $t > 0$ and $\lim_{n\to\infty} \phi^n = 0$.

(iii) **N** is the set of all positive integers .

The paper begins with the necessary definitions that are used to prove the theorems.

Definition 1. 2-metric space :

Let X be a set consisting of aleast three points. A **2-metric** on X is a mapping d from XxXxX to the set of non-negative real numbers that satisfies the following conditions:

(i) There exists three points x, y, z such that $d(x, y, z) \neq 0$,

(ii) $d(x, y, z) = 0$ if atleast two of the three points are equal ,

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$ for all x, y, z in X,

(iv) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all x, y, z, a in X

The pair (X,d) is called a 2-metric space .

Definition 2

A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be **convergent** with lim x in X if,

$$\lim_{n\to\infty} d(x_n, x, a) = 0 \quad \text{for each } a \in X.$$

Definition 3

A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be **Cauchy sequence** if,

$$\lim_{m,\, n\to\infty} d(x_n, x_m, a) = 0 \quad \text{for all } a \in X,\ m, n \in N .$$

Definition 4

A 2-metric space (X,d) is said to be **complete** if every Cauchy sequence is convergent in X.

Definition 5

A mapping T of X into itself is said to be **continuous** at a point $x \in X$, if whenever a sequence $\{x_n\}$ of X converges to $x \in X$, then the sequence $\{Tx_n\}$ converges to Tx.

Definition 6

The function ϕ is called **upper semi-continuous** if,

$$\limsup_{x \to a} \phi(x) \le \phi(a) \quad \text{for all } a \in R^+$$

and ϕ is called **lower semi-continuous** if,

$$\liminf_{x \to a} \phi(x) \ge \phi(a) \quad \text{for all } a \in R^+ .$$

Definition 7

Two mappings S, T : X → X are said to be **commuting** if ,

$$(ST)(x) = (TS)(x) \quad \text{for each } x \in X.$$

The first fixed point theorem is for for two continuous self – maps S, T commuting with an another self – map B .

Theorem 1

Let S, T be two continuous self-mappings on a complete 2-metric space (X,d) into itself satisfying the following conditions:

(1.1) ϕ is lower semi continuous ,

(1.2) $\lim_{n \to \infty} \phi^n(t) = 0$,

(1.3) S and T are surjective ,

(1.4) S and T commutes with an another the self – map B and

(1.5) $d(Bx, By, a) \ge \phi\,[\max\{d(Sx, Ty, a), d(Bx, Sx, a), d(By, Ty, a),$

$(d(Bx, Ty, a)\, d(By, Sx, a))\}]$ for all $x, y, a \in X$.

ISSN: 2278-0181
Vol. 2 Issue 3, March - 2013

Then S, T and B have a common fixed point.

Proof:

Since the mappings S and T commutes with the map B we have,

$(SB)(x) = (BS)(x)$

$(TB)(x) = (BT)(x)$ for each $x \in X$.

Now, we find an arbitrary point x_o in X such that ,

$$Bx_o = Sx_1 \quad \text{and}$$

$$Bx_1 = Tx_2 \quad \text{for } x_1, x_2 \in X.$$

Since S and T are surjective, there exists $y_o, y_1 \in X$ such that,

$$y_o = Sx_1 \quad \text{and}$$

$$y_1 = Tx_2$$

Therefore,

$$y_o = Sx_1 = Bx_o \ \text{ and}$$

$$y_1 = Tx_2 = Bx_1$$

In general,

$$\left.\begin{aligned} y_n &= Sx_{n+1} = Bx_n \quad n \in N \cup \{0\} \text{ and} \\ y_{n+1} &= Tx_{n+2} = Bx_{n+1} \quad n \in N \end{aligned}\right\} \qquad \textbf{(I)}$$

This theorem can be proved by proving,

(T.1.a) $\{y_n\}$ and $\{y_{n+1}\}$ converges to any point say 'z' in X as $n \to \infty$,

(T.1.b) $Sz = z$,

(T.1.c) $Tz = z$ and

(T.1.d) $Bz = z$.

Proof of (T.1.a) :

Taking $x = x_1$, $y = x_2$ in condition (1.5),

$$d(Bx_1, Bx_2, a) \geq \phi\,[\max\{d(Sx_1, Tx_2, a), d(Bx_1, Sx_1, a), d(Bx_2, Tx_2, a),$$

$(d(Bx_1, Tx_2, a)\ d(Bx_2, Sx_1, a))\}]$

Therefore, $d(y_1, y_2, a) \geq \phi\,[\max\{d(y_o, y_1, a), d(y_1, y_o, a), d(y_2, y_1, a),$

$(d(y_1, y_1, a)\ d(y_2, y_o, a))\}]$ **[by (I)]**

We know that $d(y_1, y_1, a) = 0$ [by condition (ii) of Definition 1].

That is, $d(y_1, y_2, a) \geq \phi\,[\max\{d(y_o, y_1, a), d(y_1, y_o, a), d(y_2, y_1, a), 0\}]$

$= \phi\,[\max\{d(y_1, y_o, a), d(y_2, y_1, a)\}]$

Suppose that, $d(y_2, y_1, a) > d(y_1, y_o, a)$

Then, $d(y_1, y_2, a) \geq \phi\,[d(y_1, y_2, a)]$. This is a contradiction.

Therefore, $d(y_2, y_1, a) \geq \phi\,[d(y_o, y_1, a)]$ **(II)**

Again taking $x = x_3$, $y = x_2$ in condition (1.5),

$$d(Bx_3, Bx_2, a) \geq \phi\,[\max\{d(Sx_3, Tx_2, a), d(Bx_3, Sx_3, a), d(Bx_2, Tx_2, a),$$

$$(d(Bx_3, Tx_2, a)\ d(Bx_2, Sx_3, a))\}]$$

This implies, $d(y_3, y_2, a) \geq \phi\,[\max\{d(y_2, y_1, a), d(y_3, y_2, a), d(y_2, y_1, a),$

$(d(y_3, y_1, a)\ d(y_2, y_2, a))\}]$ **[by (I)]**

$= \phi\,[\max\{d(y_2, y_1, a), d(y_3, y_2, a), d(y_2, y_1, a), 0\}]$

$= \phi\,[\max\{d(y_2, y_1, a), d(y_3, y_2, a)\}]$

Therefore, $d(y_3, y_2, a) \geq \phi\,[d(y_2, y_1, a)]$ **(III)**

Similarly for $x = x_3$, $y = x_4$ the condition (1.5) becomes,

$$d(Bx_3, Bx_4, a) \geq \phi\,[\max\{d(Sx_3, Tx_4, a), d(Bx_3, Sx_3, a), d(Bx_4, Tx_4, a),$$

$$(d(Bx_3, Tx_4, a)\ d(Bx_4, Sx_3, a))\}]$$

This implies, $d(y_3, y_4, a) \geq \phi\,[\max\{d(y_2, y_3, a), d(y_3, y_2, a), d(y_4, y_3, a),$

$(d(y_3, y_3, a)\ d(y_4, y_2, a))\}]$ **[by (I)]**

$= \phi\,[\max\{d(y_3, y_2, a), d(y_4, y_3, a), 0\}]$

Therefore $d(y_4, y_3, a) \geq \phi\,[d(y_2, y_3, a)]$ **(IV)**

By **(IV),** $d(y_4, y_3, a) \geq \phi\,[d(y_2, y_3, a)]$

$\geq \phi\,[\phi\, d(y_1, y_2, a)]$ [using **(III)**]

$= \phi^2\, d(y_1, y_2, a)$

$\geq \phi^2\,[\phi\, d(y_1, y_2, a)]$ [using **(II)]**

$= \phi^3\, d(y_1, y_o, a)$

Continuing the process,

$$d(y_n, y_{n+1}, a) \geq \phi^n\, d(y_1, y_o, a)$$

Taking limit as $n \rightarrow \infty$,

$\lim_{n\to\infty} d(y_n, y_{n+1}, a) = 0$ for $m>n$. [since $\lim_{n\to\infty} \phi^n = 0$]

Therefore, $\{y_n\}$ is a Cauchy sequence in X.

Since (X,d) is complete $\{y_n\}$ converges to any point say $z \in X$ as $n \rightarrow \infty$.

Consequently, the sequence $\{y_{n+1}\}$ also converges to $z \in X$ as $n \rightarrow \infty$.

That is,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = z.$$

Proof of (T.1.b) :

Taking limit as $n \rightarrow \infty$ in (I) and using (T.1.a),

$$\left.\begin{aligned} \lim_{n\to\infty} y_n &= z = \lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} Bx_n \\ \lim_{n\to\infty} y_{n+1} &= z = \lim_{n\to\infty} Tx_{n+2} = \lim_{n\to\infty} Bx_{n+1} \end{aligned}\right\} \quad (V)$$

Now setting $x = S_{Xn}$, $y = x_{n+1}$ in condition 1.5,

$$d(BSx_n, Bx_{n+1}, a) \geq \phi\,[\max\{d(S^2x_n, Tx_{n+1}, a), d(BSx_n, S^2x_n, a), d(Bx_{n+1}, Tx_{n+1}, a),$$

$$(d(BSx_n, Tx_{n+1}, a)\, d(Bx_{n+1}, S^2x_n, a))\}]$$

Since S is continuous and commutes with B, then $\{SSx_n\}$ and $\{BSx_n\}$ converges to Sz as $n \to \infty$.

Now taking limit as $n \to \infty$ to the inequality and also using **(V) ,**

$$d(Sz, z, a) \geq \phi\,[\max\{d(Sz, z, a), d(Sz, Sz, a), d(z, z, a), (d(Sz, z, a)\, d(z, Sz, a))\}]$$

$$= \phi\,[\max\{d(Sz, z, a), 0, 0, d^2(Sz, z, a), \}] \qquad \text{[by Definition 1.]}$$

$$= \phi\,[\, d^2 (Sz, z, a)\,]$$

$\therefore$ $d(Sz, z, a) \geq \phi\,[\, d^2 (Sz, z, a)\,]$. This is a contradiction.

This implies, $d(Sz, z, a) = 0 \quad \forall \quad a \in X$. Hence $Sz = z$.

Proof of (T.1.c) :

Taking $x = x_n$, $y = Tx_{n+1}$ in condition (1.5),

$$d(Bx_n, BTx_{n+1}, a) \geq \phi\,[\max\{d(Sx_n, TTx_{n+1}, a), d(Bx_n, Sx_n, a),$$

$$d(BTx_{n+1}, TTx_{n+1}, a),$$

$$(d(Bx_n, TTx_{n+1}, a)\, d(BTx_{n+1}, Sx_n, a))\}]$$

Since T is continuous and commutes with B, then $\{TTx_{n+1}\}$ and $\{BTx_{n+1}\}$ converges to Tz as $n \to \infty$.

Now taking limit as $n \to \infty$ and also using **(V) ,**

$$d(z, Tz, a) \geq \phi\,[\max\{d(z, Tz, a), d(z, z, a), d(Tz, Tz, a), (d(z, Tz, a)\, d(Tz, z, a))\}]$$

$$= \phi\,[\max\{\, d(Tz, z, a), 0, 0, d^2(Tz, z, a)\}] \qquad \text{[by Definition 1.]}$$

$$= \phi\,[d^2 (Tz, z, a)]$$

$d(z, Tz, a) \geq \phi\,[d^2 (Tz, z, a)]$ which is a contradiction.

Therefore, $d(Tz, z, a) = 0 \quad \forall \quad a \in X$. Hence $Tz = z$.

Proof of (T.1.d) :

Consider,

$$d(Bz, Bx_{n+1}, a) \geq \phi[\max \{d(Sz, Tx_{n+1}, a), d(Bz, Sx_{n+1}, a), d(Bx_{n+1}, Tx_{n+1}, a), (d(Bz, Tx_{n+1}, a)\, d(Bx_{n+1}, Sz, a))\}] \quad \text{[by condition (1.5)]}$$

Taking limit as $n \to \infty$,

$$d(Bz, z, a) \geq \phi\, [\max\{d(Sz, z, a), d(Bz, z, a), d(z, z, a), (d(Bz, z, a)\, d(z, Sz, a))\}] \quad \textbf{[by (V)]}$$

$$= \phi\, [\max\{d(z, z, a), d(Bz, z, a), d(z, z, a), (d(Bz, z, a)\, d(z, z, a))\}] \quad \text{[by (T.1.b)]}$$

$$= \phi\, [\max \{0, d(Bz, z, a), 0, 0\}] \quad \text{[by Definition 1.]}$$

$$= \phi\, [d(Bz, z, a)]$$

That is, $d(Bz, z, a) \geq \phi\, [d(Bz, z, a)]$. This is a contradiction.

Therefore, $d(Bz, z, a) = 0 \quad \forall \quad a \in X$. Hence $Bz = z$.

Therefore, z is the common fixed point of S, T and B.

The second fixed point theorem is for two continuous self-maps S and T commuting with other two self-maps A and B respectively.

Theorem 2

Let S and T be two continuous self-mappings defined on a complete 2-metric space (X,d) into itself which satisfies the following conditions :

(2.1) ϕ is upper semi-continuous,

(2.2) S and T commutes with other two self-maps A and B respectively and

(2.3) $d(Ax, By, a) \leq \phi\, [\max\{d(Sx, Ty, a), d(Sx, Ax, a), d(Sx, By, a),$

d(Ty, By, a), d(Ty, Ax, a), (d(Sx, Bx, a) d(Ty, Ay,a))}]

for all x, y, a $\in$ X then S, T, A and B have a common fixed point which are unique.

Proof:

Since the map S commutes with A and T commutes with B we have,

(SA) (x) = (AS) (x)

(TB) (x) = (BT) (x) for each $x \in X$.

Now, for an arbitrary point in x_0 in X there exists a point $x_1 \in X$ such that,

$$Sx_1 = Ax_o \quad \text{and}$$

for $x_2 \in X$, $Tx_2 = Bx_1$.

We define two sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ in X such that,

$$\left.\begin{aligned} y_{2n+1} &= Sx_{2n+1} = Ax_{2n} && n \in N \cup \{0\} \quad \text{and} \\ y_{2n} &= Tx_{2n} = Bx_{2n-1} && n \in N \end{aligned}\right\} \qquad \textbf{(I)}$$

This theorem can be proved by proving,

(T.2.a) (X,d) is bounded ,

(T.2.b) The sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$converges to any point say $z \in X$ as $n \to \infty$,

(T.2.c) Sz = z ,

(T.2.d) Tz = z ,

(T.2.e) Az = z ,

(T.2.f) Bz = z and

(T.2.g) S, T, A and B have a unique fixed point z .

Proof of (T.2.a) :

Let λ_{2r} is a sequence of real number where $0 < \lambda < 1$, $r \in N$.

First, let λ_{2r} be an increasing sequence. Suppose if $\{y_{2n}\}$ is not bounded then,

$$d(y_{2n+1}, y_{2n+2}, a) \geq \lambda_{2r} \qquad \textbf{(II)}$$

Now consider,

$d(y_{2n+1}, y_{2n+2}, y_{2n}) = d(Ax_{2n}, Bx_{2n+1}, y_{2n})$ **[using (I)]**

$\leq \phi$ [max{$d(Sx_{2n}, Tx_{2n+1}, y_{2n})$, $d(Sx_{2n}, Ax_{2n}, y_{2n})$, $d(Sx_{2n}, Bx_{2n+1}, y_{2n})$,

$d(Tx_{2n+1}, Bx_{2n+1}, y_{2n})$,$d(Tx_{2n+1},Ax_{2n},y_{2n})$,

($d(Sx_{2n}, Bx_{2n}, y_{2n})\, d(Tx_{2n+1}, Ax_{2n+1}, y_{2n})$)}] [by condition (2.3)]

$= \phi$ [max{$d(y_{2n}, y_{2n+1}, y_{2n})$, $d(y_{2n}, y_{2n+1}, y_{2n})$, $d(y_{2n}, y_{2n+2}, y_{2n})$,

$d(y_{2n+1}, y_{2n+2}, y_{2n})$, $d(y_{2n+1}, y_{2n+1}, y_{2n})$,

($d(y_{2n}, y_{2n+1}, y_{2n})\, d(y_{2n+1}, y_{2n+2}, y_{2n})$)}] [by (**I**)]

$= \phi$ [max {0, 0, 0, $d(y_{2n+1}, y_{2n+2}, y_{2n})$, 0, 0}] [by Definition 1.]

$= \phi\, [d(y_{2n}, y_{2n+1}, y_{2n+2})]$

$< d(y_{2n}, y_{2n+1}, y_{2n+2})$ [since $\phi(t)<t$]

This implies, $d(y_{2n}, y_{2n+1}, y_{2n+2}) = 0$.

Hence $d(y_{2n}, y_{2n+1}, y_{2n+2}) \leq \lambda_{2n}$. [since $0 < \lambda < 1$]

This is a contradiction to the assumption **(II)**.

Therefore, $\{y_{2n}\}$ is a bounded sequence in X.

Similarly $\{ y_{2n}\}$ is a bounded sequence in X when λ_{2r} is a decreasing sequence.

Hence (X,d) is bounded.

Proof of (T.2.b) :

Let p, m be two positive integers belonging to N.

Let the bound of $d(y_{2n}, y_{2n+1}, y_{2n+p})$ be $\phi^{2n}(\lambda_{2n})$.

That is,

$$\left.\begin{array}{l} d(y_{2n}, y_{2n+1}, y_{2n+p}) \leq \phi^{2n}(\lambda_{2n}) \quad \text{and} \\ d(y_{2n}, y_{2n+1}, y_{2n+p+m}) \leq \phi^{2n}(\lambda_{2n}) \end{array}\right\} \qquad \textbf{(III)}$$

Consider,

$d(y_{2n}, y_{2n+p}, y_{2n+p+m}) \leq d(y_{2n}, y_{2n+1}, y_{2n+p+m}) + d(y_{2n}, y_{2n+p}, y_{2n+1}) + d(y_{2n+1}, y_{2n+p}, y_{2n+p+m})$

[by Definition 1.]

$$\leq \phi^{2n}(\lambda_{2n}) + \phi^{2n}(\lambda_{2n}) + d(y_{2n+1}, y_{2n+p}, y_{2n+p+m}) \quad \text{[by } \mathbf{(III)}\text{]}$$

$$= 2\,\phi^{2n}(\lambda_{2n}) + d(y_{2n+1}, y_{2n+p}, y_{2n+p+m})$$

$$\leq 2\,\phi^{2n}(\lambda_{2n}) + d(y_{2n+1}, y_{2n+2}, y_{2n+p+m}) + d(y_{2n+1}, y_{2n+p}, y_{2n+2})$$

$$+ d(y_{2n+2}, y_{2n+p}, y_{2n+p+m}) \quad \text{[by Definition 1.]}$$

$$\leq 2\,\phi^{2n}(\lambda_{2n}) + \phi^{2n+1}(\lambda_{2n}) + \phi^{2n+1}(\lambda_{2n}) + d(y_{2n+2}, y_{2n+p}, y_{2n+p+m})$$

[by **(III)**]

$$= 2\,\phi^{2n}(\lambda_{2n}) + 2\,\phi^{2n+1}(\lambda_{2n}) + d(y_{2n+2}, y_{2n+p}, y_{2n+p+m})$$

$$= 2\,[\phi^{2n}(\lambda_{2n}) + \phi^{2n+1}(\lambda_{2n})] + d(y_{2n+2}, y_{2n+p}, y_{2n+p+m})$$

Continuing the process,

$$d(y_{2n}, y_{2n+p}, y_{2n+p+m}) \leq 2 \sum_{i=2n}^{2n+p-1} \phi^{i}(\lambda_{2n}) \qquad \textbf{(IV)}$$

Consider,

$$d(Ax_{2n+m}, Bx_{2n+m+p}, y_{2n+m}) \leq \phi\,[\max\{d(Sx_{2n+m}, Tx_{2n+m+p}, y_{2n+m}),$$
$$d(Sx_{2n+m}, Ax_{2n+m}, y_{2n+m}), d(Sx_{2n+m}, Bx_{2n+m+p}, y_{2n+m}),$$
$$d(Tx_{2n+m+p}, Bx_{2n+m+p}, y_{2n+m}), d(Tx_{2n+m+p}, Ax_{2n+m}, y_{2n+m}),$$
$$(d(Sx_{2n+m}, Bx_{2n+m}, y_{2n+m})\, d(Tx_{2n+p+m}, Ax_{2n+m+p}, y_{2n+m}))\}]$$

[by condition (2.3)]

Therefore,

$$d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}) \leq \phi\,[\max\{d(y_{2n+m}, y_{2n+m+p}, y_{2n+m}), d(y_{2n+m}, y_{2n+m+1}, y_{2n+m}),$$
$$d(y_{2n+m}, y_{2n+m+p}, y_{2n+m}), d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}),$$
$$d(y_{2n+m+p}, y_{2n+m+1}, y_{2n+m}),$$
$$(d(y_{2n+m}, y_{2n+1+m}, y_{2n+m})\, d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}))\}]$$

[by **(I)**]

$$= \phi\,[\max\{0, 0, 0, d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}), d(y_{2n+m+p}, y_{2n+m+1}, y_{2n+m}), 0\}] \quad \text{[by Definition 1.]}$$

$$= \phi\,[\max\{ d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}), d(y_{2n+m+p}, y_{2n+m+1}, y_{2n+m})\}$$

Suppose that,

$$d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}) > d(y_{2n+m+p}, y_{2n+m+1}, y_{2n+m})$$

Then,

$$d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}) \le \phi\,[d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m})]$$

$$< d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}) \qquad \text{[since } \phi(t)<t]$$

This is a contradiction. Therefore,

$$d(y_{2n+m+p}, y_{2n+m+p+1}, y_{2n+m}) \le \phi\,[d(y_{2n+m+p}, y_{2n+m+1}, y_{2n+m})\}]$$

$$\le \phi\,[2 \sum_{i=2n}^{2n+p+m-1} \phi^{i}(\lambda_{2n})] \qquad \text{[by } \mathbf{(IV)}]$$

$$= 2 \sum_{i=2n}^{2n+p+m-1} \phi^{i+1}(\lambda_{2n})$$

$$= 2\,[\phi^{2n+1} + \phi^{2n+2} + \ldots + \phi^{2n+p+m}]$$

Since p, m is finite and allowing limit as $n \to \infty$,

$$d(y_{2n+m}, y_{2n+m+p+1}, y_{2n+p+m}) = 0 \ . \qquad [\text{since } \lim_{n\to\infty} \phi^{n} = 0]$$

Hence $\{y_{2n+m}\}$ is a Cauchy sequence for $m \in N$.

Therefore, $\{y_{2n}\}$ is also a Cauchy sequence in X.

Since (X,d) is complete, $\{y_{2n}\}$ converges to point $z \in X$ as $n \to \infty$.

The sequence $\{y_{2n+1}\}$ also converges to $z \in X$ as $n \to \infty$.

That is, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = z.$

Proof of (T.2.c) :

Now taking limit as $n \to \infty$ in **(I)** and using **(T.2.b),**

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} y_{2n+1} = z = \lim\limits_{n\to\infty} Sx_{2n+1} = \lim\limits_{n\to\infty} Ax_{2n} \quad n\in N \cup \{0\} \quad \text{and} \\ \lim\limits_{n\to\infty} y_{2n} = z = \lim\limits_{n\to\infty} Tx_{2n} = \lim\limits_{n\to\infty} Bx_{2n-1} \quad n\in N \end{array}\right\} \qquad \textbf{(V)}$$

Consider,

$$d(ASx_{2n}, Bx_{2n+1}, a) \le \phi\,[\max\{ d(S^2x_{2n}, Tx_{2n+1}, a), d(S^2x_{2n}, ASx_{2n}, a), d(S^2x_{2n}, Bx_{2n+1}, a),$$
$$d(Tx_{2n+1}, Bx_{2n+1}, a), d(Tx_{2n+1}, ASx_{2n}, a),$$
$$(d(S^2x_{2n}, BSx_{2n}, a)\, d(Tx_{2n+1}, Ax_{2n+1}, a))\}] \quad \text{[by condition (2.3)]}$$

Since S is continuous and S commutes with A the sequence $\{SSx_{2n}\}$ and $\{ASx_{2n}\}$ converges to the point Sz as $n \to \infty$.

Now, taking limit as $n \to \infty$ and also using **(V),**

$$d(Sz, z, a) \le \phi\,[\max\{d(Sz, z, a), d(Sz, Sz, a), d(Sz, z, a), d(z, z, a), d(z, Sz, a),$$
$$(d(Sz, Bz, a)\, d(z, z, a))\}]$$
$$= \phi\,[\max\{d(Sz, z, a), 0, d(Sz, z, a), 0, d(z, Sz, a), 0\}]$$
[by Definition 1.]
$$= \phi\,[d(Sz, z, a)]$$

$d(Sz, z, a) \le \phi\,[d(Sz, z, a)] < d(Sz, z, a)$ [since $\phi(t)<t$]

This is a contradiction.

This implies $d(Sz, z, a) = 0 \quad \forall\; a\in X$. Therefore, $Sz = z$.

Proof of (T.2.d) :

Consider,

$$d(Ax_{2n}, BTx_{2n+1}, a) \le \phi\,[\max\{ d(Sx_{2n}, T^2 x_{2n+1}, a), d(Sx_{2n}, Ax_{2n}, a), d(Sx_{2n}, BTx_{2n+1}, a),$$
$$d(T^2x_{2n+1}, BTx_{2n+1}, a), d(T^2x_{2n+1}, Ax_{2n}, a),$$
$$(d(Sx_{2n}, Bx_{2n}, a), d(T^2 x_{2n+1}, ATx_{2n+1}, a))\}] \quad \text{[by condition(2.3)]}$$

Since T is continuous and B commutes with T, then the sequences$\{TTx_{n+1}\}$ and $\{BTx_{n+1}\}$ converges to Tz as $n \to \infty$.

Taking limit as $n \to \infty$ and also using **(V),**

$$d(z, Tz, a) \leq \phi\,[\max\{d(z, Tz, a), d(z, z, a), d(z, Tz, a), d(Tz, Tz, a), d(Tz, z, a), d((z, z, a)\; d(Tz, Az, a))\}]$$

$$= \phi\,[\max\{d(z, Tz, a), 0, d(z,Tz, a), 0, d(Tz, z, a), 0\}]$$

[by Definition 1.]

$$= \phi\,[d(z, Tz, a)]$$

$d(z, Tz, a) \leq \phi\ [d(z, Tz, a)] < d(z, Tz, a)$ [since $\phi(t)<t$]

This is a contradiction.

This implies $d(z, Tz, a) = 0\ \ \forall\ a \in X$. Therefore, $Tz = z$.

Proof of (T.2.e) :

Consider,

$$d(Az, Bx_{2n+1}, a) \leq \phi\,[\max\{d(Sz, Tx_{2n+1}, a), d(Sz, Az, a), d(Sz, Bx_{2n+1}, a), d(Tx_{2n+1}, Bx_{2n+1}, a), d(Tx_{2n+1}, Az, a), (d(Sz, Bz, a), d(Sx_{2n+1}, Ax_{2n+1}, a))\}]$$ [by condition (2.3)]

Taking limit as $n \to \infty$,

$$d(Az, z, a) \leq \phi\,[\max\{d(z, z, a), d(z, Az, a), d(z, z, a), d(z, z, a), d(z, Az, a), (d(z, Bz, a)\; d(z, z, a))\}]$$ [by **(V)** and **(T.2.c)**]

$$= \phi\,[\max\{0, d(z, Az, a), 0, 0, d(z, Az, a), 0\}]$$ [by Definition 1.]

$$= \phi\,[d(z, Az, a)]$$

$d(Az, z, a) \leq \phi\ [d(z, Az, a)] < d(z, Az, a)$ [since $\phi(t)<t$]

This is a contradiction.

This implies $d(z, Az, a) = 0\ \forall\ a \in X$.

Therefore, $Az = z$.

Proof of (T.2.f) :

Consider,

$$d(Ax_{2n}, Bz, a) \le \phi\,[\max\{\, d(Sx_{2n}, Tz, a), d(Sx_{2n}, Ax_{2n}, a),$$
$$d(Sx_{2n}, Bz, a),\ d(Tz, Bz, a), d(Tz, Ax_{2n}, a),$$
$$(d(Sx_{2n}, Bx_{2n}, a)\, d(Tz, Az, a))\}] \quad \text{[by condition (2.3)]}$$

Taking limit as $n \to \infty$,

$$d(z, Bz, a) \le \phi\,[\max\{d(z, z, a), d(z, z, a), d(z, Bz, a), d(z, Bz, a),$$
$$d(z, z, a), (d(z, z, a)\, d(z, Az, a))\}] \quad \text{[by } \mathbf{(V)} \text{ and } \mathbf{(T.2.d)}\text{]}$$
$$= \phi\,[\max\{0, 0, d(z, Bz, a), d(z, Bz, a), 0, 0\}] \quad \text{[by Definition 1.]}$$
$$= \phi\,[d(z, Bz, a)]$$

$d(z, Bz, a) \le \phi\,[d(z, Bz, a)] < d(z, Bz, a)$ [since $\phi(t)<t$]

This is a contradiction.

This implies, $d(z, Bz, a) = 0\ \ \forall\ a\in X$. Therefore, $Bz = z$.

Proof of (T.2.g) :

If possible let z is not unique, let w be another fixed point.

$$\left.\begin{aligned} \text{Then,}\quad & z = Az = Bz = Sz = Tz \quad \text{and} \\ & w = Aw = Bw = Sw = Tw \end{aligned}\right\} \qquad \mathbf{(\ VI\ )}$$

Consider,

$$d(Aw, Bz, a) \le \phi\,[\max\{\, d(Sw, Tz, a), d(Sw, Aw, a), d(Sw, Bz, a),$$
$$d(Tz, Bz, a),\ d(Tz, Aw, a),$$
$$(d(Sw, Bw, a)\, d(Tz, Az, a))\}] \quad \text{[by condition (2.3)]}$$

Therefore, $d(w, z, a) \le \phi\,[\max\{d(w, z, a), d(w, w, a), d(w, z, a), d(z, z, a), d(z, w, a),$

$$(d(w, w, a)\, d(z, z, a))\}] \quad \text{[by } \mathbf{(\ VI\ )}\text{]}$$
$$= \phi[\max\{d(w, z, a), 0, d(w, z, a), 0, d(z, w, a), 0\}] \quad \text{[by Definition 1.]}$$
$$= \phi\,[d(z, w, a)]$$

$d(w, z, a) \le \phi\,[d(z, w, a)] < d(z, w, a)$ [since $\phi(t)<t$]

This is a contradiction.

This implies, $d(w, z, a) = 0 \quad \forall \; a \in X$. Hence $w = z$.

Therefore, z is the unique common fixed point of S, T, A and B.

CONCLUSION

While observing both the theorems we analyse that the property of a self-map, the continuity nature of the map is clearly exposed. Further commutativity plays an essential role in obtaining fixed point and its part is more dominant compared to the other two. In other words the concept of continuity and the property of commutativity gets binded to obtain a fixed point. To conclude we can say that the notion of continuity and commutativity is co-related and both plays an effective part for the existence of the fixed point.

REFERENCES :

1. S. Gahler, 2-metrische Raume and ihre topologische struktur, *Math. Nachr.* 26, 115-148, (1963).
2. -- Espaces 2-metriques et localement 2-metriques, *Ann. Scient. Ec. Norm. Sup*. 82, 387-396, (1965)
3. --. Uber die Uniformisierbarkeit 2-metrischer Raume, *Math. Nachr*. 28, 235-244 (1965).
4. --.Uber eine Zwischen relation in 2-metrischen Raumen, *Math. Nachr*. 29, 217-235, (1965)
5. G.P. Murphy, Convexity and embedding in a class of 2-metrics, Diss., Saint Louis University, (1966).
6. R. Freese and E. Andlafte, A Characterization of 2-Betweeness in 2-Metric spaces, *Canad. J. Math*. 18, 963-968, (1966).
7. S. Gahler , Zur Goemetrie 2-metrischer Raume, Revue Roumaine de Mathem. Pures et Appliquees XI, 665-667 (1966).
8. E. Andalafte and R. Freese, Existence of 2-segments in 2-metric spaces, *Fund. Math.* LX. 201-208 (1967).

9. K.E. Grant, A 2-metric lattice structure, Doctoral Diss. Saint Louis University, (1968).
10. A. Miller, 2-modular lattices and 2-metric spaces, *J. Natur. Sci. Math.* 13, 1-10 (1973)
11. R. Freese, A 2-metric characterization of the Euclidean Plane, *Math. Ann.* 206, 285-294, 1973
12. K. Iseki, Fixed point theorems in 2-metric spaces, *Math. Seminar Notes*, Kobe Univ., 3(1975), 133-136.
13. K. Iseki, A Property of orbitrally continuous mappings on 2-metric spaces, *Mathematics Seminar Notes*, Kobe Univ. 3 (1975)
14. K. Iseki, Fixed point theorem in 2-metri spaces, *Mathematics Seminar Notes*, Kobe Univ. 3 (1975)
15. K. Iseki, Fixed point theorems in Banach spaces, *Math. Seminar Notes*, Kobe Univ., 2 (1976), 11-13.
16. K. Iseki, P. L. Sharma and B. K. Sharma, Contraction type mapping on 2-metric spaces, *Math. Japonicae*, 21(1976), 67-70.
17. C.H.Diminie and A. White, Non-expansive Mappings in Linear 2-Normed Sapces, *Math. Japonicae*, 21, 197-200, (1976).
18. K.Iseki ,P.L.Sharma and B.K.Sharma, Contraction type mapping on a 2 – metric space *Math.Jap.*21, 67 -70 (1976)
19. B.E.Rhoades, Contraction type mappings on a 2–metric space, *Math. Nachr.* 91, 451-455, (1979).
20. S.L.Singh, Some contractive type principles on 2– metric spaces and applications. *Math.Seminar Notes* 7 (1), 1 – 11 (1979)
21. Ashok K.Sharma A note on fixed points in 2-metric spaces, *Indian J.Pure.Appl.* 11(12): 1580-1583 Dec. (1980).
22. --. 1980(a) Fixed points in Contractive mappings in 2-metric space (communicated)
23. --. 1980(b) Fixed point theorem in 2-metric spaces (communicated)

24. S.L. Singh, B.M.L.Tiwari and V.K.Gupta, Common fixed point theorems of commuting mappings in 2-metric spaces and an application *Math. Nachr.*95, 293-297 (1980).
25. M.S.Khan and Brain Fisher, Some fixed point theorems for commuting mappings *Math.Nachr*.106 (1982) 323-326.
26. S.V.R.Naidu and J.Rajendra Prasad Fixed point theorems in 2-metric space *Indian J.Pure.Appl. 17(8)*,974 – 993 (1986)
27. Ganguly ,Ashok and Chandel R.S Common fixed point theorems of mappings on 2-metric spaces and its applications *J.Indian Acad.Math*. 9 , 1, 26 (1987)
28. M. Imdad, M. S. Khan, M. D. khan, A common fixed point theorem in 2-metric spaces. Math. Japon. 36 (1991), no. 5, 907-914.
29. M.S.Rathore and Usha Rani, Common fixed point theorems for weakly commuting mappings in 2-metric space, *Bull. Cal. Math. Soc*. 89, 253-266 (1997).
30. Mashadi, Abu Osman, Md. T. Bin, Fixed point theorems in 2-metric spaces. (Malayan) *Bull. Malaysian Math. Soc.* (2) 22 (1999), no. 1, 11-22.
31. Singh B. and R.K.Sharma, Fixed points for expansion maps in 2-metric space *Acta Ciencia Indica* XXVIM 23 -26 (2000)
32. S. V. R. Naidu, Some fixed point theorems in metric and 2-metric spaces. *Int. J. Math. Sci.* 28 (2001), no. 11, 625-636.
33. D. Tan, Z. Liu, J. K. Kim, Common fixed points for compatible mappings of type (P) in 2-metric spaces. Nonlinear Funct. *Anal. Appl.* 8 (2003), no. 2, 215-232.
34. Z. Liu, C. Feng, S. Chun, Fixed and periodic point theorems in 2-metric spaces. Nonlinear Funct. *Anal. Appl.* 8 (2003), no. 4, 497-505.
35. Meera Srivastava and Subal Chandra Ghosh Fixed point theorems in 2-metric space *Acta Ciencia Indica* Vol XXXM. No.3, 535 (2004).
36. L. Shambhu Singh and L. Sharmeswar Singh, Some Common Fixed Point Theorems in 2-Metric Space, *International Transactions in Mathematical Sciences and Computer*, Volume 3, No. 1, (2010), pp. 121-129