Gauss Legendre quadrature over a parabolic region

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Abstract

In this paper, we introduce a Gauss Legendre quadrature method for numerical integration over a parabolic region; $R = \{(x, y) \ / \ 0 \le x \le 1,\ 0 \le y \le x^2\}$, *using transformation of variables a general formulae for numerical integration over the region R are derived which can be directly used for integrating arbitrary function over such region the performances of the method is illustrated with several numerical examples.*

Keywords: Gauss Legendre quadrature , parabolic region, extended numerical integration

1. Introduction

Numerical integration is a very important and interesting topic of numerical methods. The main objective in Numerical integration is to develop techniques that efficiently estimate the value of definite integral. Numerical methods for integration approximate a definite integral of a given function by a weighted sum of function values at specified points. There are many quadrature methods available for approximating integrals.

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Surface integrals are used in multiple areas of physics and engineering. In particular, they are used for Problems involving calculations of mass of a shell, center of mass and moments of inertia of a shell, fluid flow and mass flow across a surface, electric charge distributed over a surface, plate bending, plane strain, heat conduction over a plate, and similar problems in other areas of engineering which are very difficult to analyse using analytical techniques, These problems can be solved using the finite element method.

From the literature review we may realize that several works in numerical integration using Gaussian quadrature over triangle region have been carried out [1-6], Generalized Gaussian quadrature rules over regions with parabolic edges given in [8], in this paper we use Gauss Legendre quarature method to evaluate the surface integral over the arbitrary function in parabolic region R.

The paper is organized as follows. In Section 2 we will introduce the Gauss Legendre quadrature formula over parabolic region and In Section 3 we compare the numerical results with some illustrative examples.

2. Formulation of integrals over a parabolic region

The Numerical integration of an arbitrary function f over the parabolic region is given by $\mathrm{I} = \iint_{\mathrm{R}} \mathrm{f(x,y)dx\,dy} = \int_0^1 \left(\int_0^{x^2} f(x,y)dy\right) dx = \int_0^1 \left(\int_0^{y^2} f(x,y)dx\right) dy$ (1)

Where R is the parabolic region bounded by $\{\ (0,0),(1,0),(0,1)\ \}$ in the xy - plane

The integral of the eqn.(1) can be transformed to the square $\{(\ u\ ,\ v\ )\ /\ 0 \leq\ u \leq 1\ ,\ 0 \leq v \leq 1\}$

mathematical transformation is

$$x = \sqrt{u} \quad and \quad y = u\,v \tag{2}$$

$$dx\,dy = \ |J|\ du\,dv$$

Where $|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\sqrt{u}}{2}$

Eqn.(1) becomes

$$\int_0^1 \int_0^{x^2} f(x,y)\ dy\,dx = \int_0^1 \int_0^1 f\left(\sqrt{u}\,,u\ v\right) \frac{\sqrt{u}}{2}\ du\,dv \tag{3}$$

The above integral can be transformed further into an integral over the standard 2- square $\{(\xi,\eta)/\ 1 \leq\ \xi,\eta \leq -1\}$ by the substitution

$u = \frac{\xi+1}{2}$ and $v = \frac{\eta+1}{2}$

$du\,dv = \ |J|\ d\xi\ d\eta = \frac{1}{4}\ d\xi\ d\eta$

Eqn.(3) becomes

$$I = \int_0^1 \int_0^{x^2} f(x,y)\ dy\,dx = \int_{-1}^1 \int_{-1}^1 f\left(\sqrt{\frac{\xi+1}{2}}\,, \frac{(\xi+1)(\eta+1)}{4}\right) \frac{1}{8} \sqrt{\frac{\xi+1}{2}}\ d\xi\ d\eta$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{1}{8} \sqrt{\frac{\xi_i+1}{2}}\ w_i\,w_j\ f(x(\xi_i\,,\eta_j)\,, y(\xi_i\,,\eta_j)) \tag{4}$$

Where $\xi_{i\,,}\,\eta_j$ are Gaussian points and w_i, w_j are corresponding weights. We can rewrite eqn.(4) as

$$\mathrm{I} = \sum_k^{N=n\times n} a_k f(x_k, y_k) \tag{5}$$

Where $a_k = \frac{1}{8}\sqrt{\frac{\xi_i+1}{2}}\, w_i\ w_j$, (5a)

$x_k = \sqrt{\frac{\xi_i+1}{2}}$ and $y_k = \frac{(\xi_i+1)(\eta_j+1)}{4}$, if $k, i, j = 1,2,3, - - - -$ (5b)

we find out new Gaussian points x_k , y_k and weights coefficients a_k of various order N=5,10,15,20 by using eqns.(5a-b) and tabulated in Table 1

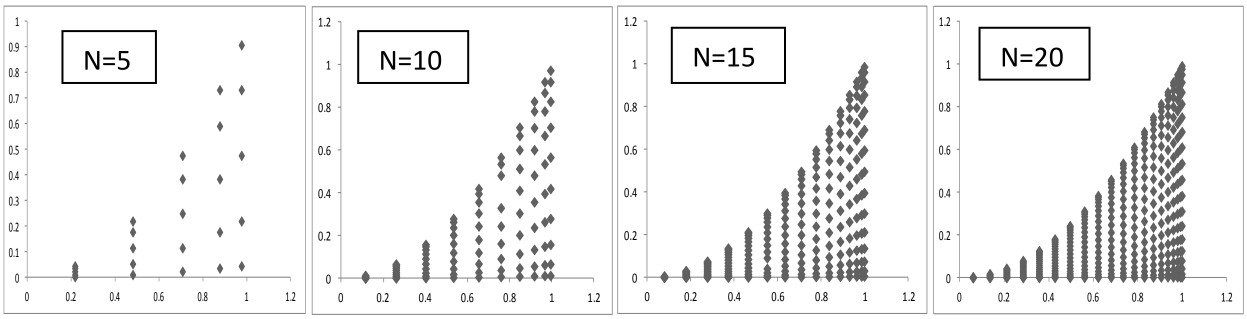

Fig. 1 Gaussian points (x_k , y_k) values for the region R

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x_k	y_k	a_k
0.2165873427	0.0022005553	0.0015197486
0.4803804169	0.0108252201	0.0068093924
0.7071067812	0.0234550385	0.0119134298
0.8770602345	0.0360848569	0.0124323288
0.9762632447	0.0447095217	0.0068502376
0.2165873427	0.0108252201	0.0030701255
0.4803804169	0.0532526444	0.0137560177
0.7071067812	0.1153826724	0.0240669566
0.8770602345	0.1775127005	0.0251152122
0.9762632447	0.2199401248	0.0138385314
0.2165873427	0.0234550385	0.0036490925
0.4803804169	0.1153826724	0.0163501397
0.7071067812	0.2500000000	0.0286055246
0.8770602345	0.3846173275	0.0298514612
0.9762632447	0.4765449614	0.0164482140
0.2165873427	0.0360848569	0.0030701255
0.4803804169	0.1775127005	0.0137560177
0.7071067812	0.3846173275	0.0240669566
0.8770602345	0.5917219545	0.0251152122
0.9762632447	0.7331497981	0.0138385314
0.2165873427	0.0447095217	0.0015197486
0.4803804169	0.2199401248	0.0068093924
0.7071067812	0.4765449614	0.0119134298
0.8770602345	0.7331497981	0.0124323288
0.9762632447	0.9083804012	0.0068502376

Table 1. Gaussian Points and weighting coefficient over the region R for N = 5

3. Numerical results

Exact value	Order	Computed value
1) $\int_0^1 \int_0^{x^2} \sqrt{x+y}\,dydx$ = 0.3354005175	N=5 N=10 N=15 N=20	0.3354780675879244 0.3360479380805889 0.3352516333883463 0.3354012131862628
2) $\int_0^1 \int_0^{x^2} \frac{xy}{\sqrt{1+x^2+y^2}}\,dydx$ = 0.0589780486	N=5 N=10 N=15 N=20	0.0589780085320245 0.0585261879099782 0.0589525800550520 0.0589780255096226
3) $\int_0^1 \int_0^{y^2} e^{y^2} \sin(xy)\,dxdy$ = 0.1099369995	N=5 N=10 N=15 N=20	0.1099369044010397 0.1093299584364246 0.1099060103873312 0.1099369584167240
4) $\int_0^1 \int_0^{x^2} \frac{\cos(x+y)}{\sqrt{1+x^2+y^2}}\,dydx$ = 0.1280603501	N=5 N=10 N=15 N=20	0.1283843822343174 0.1281956972180804 0.1282390608892068 0.1280663766589819
5) $\int_0^1 \int_0^{y^2} \frac{x^4+y^4}{1+xy^2}\,dydx$ = 0.1352910112	N=5 N=10 N=15 N=20	0.1352914109027507 0.1352889650184565 0.1351939230076843 0.1352908735411647
6) $\int_0^1 \int_0^{x^2} \sqrt{x^2+y^2}\,\sin(10x)\,dydx$ = 0.06966170118	N=5 N=10 N=15 N=20	0.0693065043956028 0.0695098671907841 0.0695026948435256 0.0696617070526936

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4. Conclusions

In this paper we have derived extended numerical integration of Gauss Legendre quadrature rule for calculating over a parabolic region $\{(x, y) \;/\; 0 \le x \le 1, 0 \le y \le x^2\}$, new Gaussian points and its weights are calculated of various order N = 5, 10,15,20. We have then demonstrated the application of the derived quadrature rule by considering the evaluation of some typical surface integrals over the region R , the results obtained are in excellent agreement with the exact value

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