Gauss Legendre quadrature over a unit circle

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Abstract

This paper presents a Gauss Legendre quadrature over a unit circle region $\left\{(x,y)\ /\ 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\right\}$, *are derived using transformation of variables new Gaussian points and corresponding weights are calculated. The numerical evaluation of the unit circle domain integrals of any arbitrary functions is illustrated with some numerical examples.*

Keywords: Finite element method, Gauss Legendre quadrature , unit circle region, extended numerical integration

1. Introduction

The finite element method is essentially a numerical method for the approximate solution of practical problems arising in engineering and scientific analysis. The advantages of the finite element technique over other alternatives are more fully appreciated in two dimensional situations. Surface integrals are used in multiple areas of physics and engineering. In particular, they are used for Problems involving calculations of mass of a shell, center of mass and moments of inertia of a shell, fluid flow and mass flow across a surface, electric charge distributed over a surface, plate bending, plane strain, heat conduction over a plate, and similar problems in other areas of engineering which are very difficult to analyse using analytical techniques, These problems can be solved using the finite element method.

From the literature review we may realize that several works in numerical integration using Gaussian quadrature over triangle region and linear convex quadrilateral region have been carried out [1-6,12]. The both Gaussian and Szego quadrature formulae depend on the location of the singularities of the integrand f(z) with respect to the unit circle given in [8-9]

The paper is organized as follows. In Section 2 we will introduce the Gauss Legendre quadrature formula over unit circle region and In Section 3 we compare the numerical results with some illustrative examples.

2. Formulation of integrals over a unit circle

The Numerical integration of an arbitrary function f over the unit circle is given by

$$\mathrm{I} = \iint_{\mathrm{A}} \mathrm{f}(\mathrm{x},\mathrm{y})\mathrm{dx}\,\mathrm{dy} = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} f(x,y)dy\right)dx = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} f(x,y)dx\right)dy \qquad (1)$$

Where A is the unit circle $\{\ (0,0),(1,0),(0,1)\ \}$ in the xy - plane

The integral of the eqn.(1) can be transformed to r$\boldsymbol{\theta}$ – plane. Transformation is

$$r = \left(\frac{b-a}{2}\right)\xi + \left(\frac{b+a}{2}\right) \tag{2}$$

$$\theta = \left(\frac{\beta-\alpha}{2}\right)\eta + \left(\frac{\beta+\alpha}{2}\right) \tag{3}$$

a = 0, b = 1 and $\alpha = 0,\ \beta = \pi/2$ eqn.(2) and eqn.(3) becomes

$$r = \frac{\xi+1}{2} \quad , \qquad \theta = (\eta+1)\pi/4$$

Transform the rectangle in to the standard square $1 \leq \xi, \eta \leq -1$. by change of variables it follows that

$$x = r\cos\theta = \left(\frac{\xi+1}{2}\right)\cos((\eta+1)\pi/4) \tag{4}$$

$$y = r\sin\theta = \left(\frac{\xi+1}{2}\right)\sin((\eta+1)\pi/4) \tag{5}$$

we have

$$\mathrm{I} = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)dxdy = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, J\, d\xi\, d\eta \tag{6}$$

Where $J(\xi,\eta)$ is the Jacobians of the transformation

$$J(\xi,\eta) = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{(\xi+1)\pi}{16}$$

From eqn. (6) , we can write as

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1} f\left(\left(\frac{\xi+1}{2}\right)\cos\left(\frac{(\eta+1)\pi}{4}\right),\ \left(\frac{\xi+1}{2}\right)\sin\left(\frac{(\eta+1)\pi}{4}\right)\right) \frac{(\xi+1)\pi}{16}\, d\xi\, d\eta$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(\xi_i+1)\pi}{16} w_i w_j\, f\left(x(\xi_i,\eta_j), y(\xi_i,\eta_j)\right) \tag{7}$$

Where ξ_i, η_j are Gaussian points and w_i, w_j are corresponding weights. We can rewrite eqn. (7) as

$$\mathrm{I} = \sum_{k}^{N=n\times n} a_k f(x_k, y_k) \tag{8}$$

Where $a_k = \frac{(\xi_i+1)\pi}{16} w_i w_j$, (8a)

$$x_k = \left(\frac{\xi_i+1}{2}\right)\cos((\eta_j+1)\pi/4) , \tag{8b}$$

$$y_k = \left(\frac{\xi_i+1}{2}\right)\sin((\eta_j+1)\pi/4) , \tag{8c}$$

if $k = 1,2,3,----$, $i,j = 1,2,3,----$

we find out new Gaussian points x_k , y_k and weights coefficients a_k of various order N =5,10,15,20 by using eqn.(8a),(8b) and(8c) and tabulated in Table 1

Table 1. Gaussian Points and weighting coefficient over the region A for N = 5

x_k	y_k	a_k
0.04678278193197457	0.00345349702889218	0.00103408138715589 1
0.23013914053182605	0.01698883232754596	0.01027645984811559 5
0.49864320092023384	0.03680975653306203	0.02646498707663774 4
0.76714726130864160	0.05663068073857810	0.03425561601651970 5
0.95050361990849350	0.07016601603723191	0.02100982586287565 0
0.04386178550298574	0.01663427484182355	0.00208900310914561 6
0.21576984525046589	0.08182920205630174	0.02076002608727668 7
0.46750920355886550	0.17729958992551350	0.05346333564580476 5
0.71924856186726520	0.27276997779472530	0.06920160178206564 0
0.89115662161474560	0.33796490500920360	0.04244307275548970 4
0.03317043357436901	0.03317043357436901	0.00248294916476144 7
0.16317574027498855	0.16317574027498850	0.02467497018465970 0
0.35355339060000000	0.35355339060000000	0.06354549881038908 0
0.54393104091155890	0.54393104091155890	0.08225170110694983 0
0.67393634761217880	0.67393634761217880	0.05044702498851407 0
0.01663427484182356	0.04386178550298574	0.00208900310914561 6
0.08182920205630176	0.21576984525046589	0.02076002608727668 7
0.17729958992551353	0.46750920355886550	0.05346333564580476 5
0.27276997779472534	0.71924856186726520	0.06920160178206564 0
0.33796490500920370	0.89115662161474560	0.04244307275548970 4
0.00345349702889215	0.04678278193197457	0.00103408138715589 1
0.01698883232754580	0.23013914053182605	0.01027645984811559 5
0.03680975653306168	0.49864320092023384	0.02646498707663774 4
0.05663068073857757	0.76714726130864160	0.03425561601651970 5
0.07016601603723124	0.95050361990849350	0.02100982586287565 0

3. Numerical results

Exact value	Order	Computed value
1) $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\,dydx$ = 0.1666666667	N=5	0.1657006169130470
	N=10	0.1665461244505496
	N=15	0.1665743377735694
	N=20	0.1666497582248984
2) $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{1+x^2+y^2}\,dydx$ = 0.5443965226	N=5	0.5443959985924995
	N=10	0.5447587157947142
	N=15	0.5445440217870158
	N=20	0.5443965510242446
3) $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{-x^2}\sin(x+y)\,dydx$ = 0.4339584397	N=5	0.4339594454705387
	N=10	0.4355817866124337
	N=15	0.4336350226254936
	N=20	0.4339585168822540
4) $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dydx$ = 0.5235987756	N=5	0.523598773172042
	N=10	0.526152432804568
	N=15	0.523273240355146
	N=20	0.523598898977512
5) $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{x^4+y^4}{1+x^2y}\,dydx$ = 0.1761129344	N=5	0.176121028788965
	N=10	0.177904823327757
	N=15	0.176094429817746
	N=20	0.176112985432682
6) $\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\sin(10x)\,dydx$ = 0.08799717224	N=5	0.093765742603105
	N=10	0.082329251719011
	N=15	0.088395981414517
	N=20	0.087997185681533

4. Conclusions

In this paper we have derived Gauss Legendre quadrature method for calculating integral over a unit circle region $\{(x, y) \ / \ 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$, new Gaussian points and its weights are calculated of various order N = 5, 10,15,20. We have then evaluate the typical integrals governed by the proposed method. The results obtained are in excellent agreement with the exact value

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