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GENERALIZATION OF CANONICAL POLYNOMIALS FOR OVERDETERMINED $m-th$ ORDER ORDINARY DIFFERENTIAL EQUATIONS(ODEs)

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ABSTRACT

Canonical polynomials play a remarkable roles in Lanczos' Recursive formulation of the tau method. Meanwhile, their construction are done for individual cases, and the problems of indeterminate ones are most of the time overwhelming, if not impossible for overdetermined cases. In this paper, we shall present a derived formula for a general class of m-th order overdetermined ODEs. As their derivatives are of equal level of imporatnce, a general formula for that is also reported in this paper. The principle of mathematical induction is employed to establish the validity of the two formulae.

1.0 INTRODUCTION

Ortiz [8] gave a step-by-step account of Lanczos [6] Tau method and its applications in solving both initial value problems (IVPs) and boundary value problems (BVPs). The essential of the Tau method (Lanczos [6] and Ortiz [8]) is to perturb the given differential problem in such a way that its exact solution becomes a polynomial. To achieve this, a polynomial perturbation term is added to the right hand side of the differential equation. The derived Tau approximation is written in terms of a special polynomial basis, uniquely associated with the given differential operator L (see Ortiz [8]) which defines the given problem. Such basis does not depend on the degree of approximation. The order of the approximation can be increased by just adding one or more canonical polynomials to those already generated and updating the coefficients affecting them.

2.0 PROBLEM STATEMENT AND METHODOLOGY

In this paper, we intend to obtain a general formula for the canonical polynomials and the derivatives of such polynomials for overdetermined m-th order initial value problems (IVPs)

$$Ly(x) := \sum_{r=0}^{m} \left\{ \sum_{k=0}^{N_r} P_{rk} x^k \right\} y^{(r)}(x) = \sum_{r=0}^{F} f_r x^r \tag{2.1a}$$

$$Ly(x) := \sum_{r=0}^{m} \left\{ \sum_{k=0}^{N_r} P_{rk} x^k \right\} y^{(r)}(x) = \sum_{r=0}^{F} f_r x^r \tag{2.1a}$$

$$L^* y(x_{rk}) := \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \qquad k = 1(1)m \tag{2.1b}$$

where N_r, F are given non-negative integers and a_{rk}, x_{rk}, α_k, f_r, P_{rk} are given real numbers by seeking an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \qquad n < +\infty \tag{2.2}$$

which is the exact solution of the corresponding perturbed problem

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + H_n(x) \tag{2.3a}$$

$$L^* y_n(x_{rk}) = \alpha_k, \;\; k = 1(1)m \tag{2.3b}$$

where

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \tag{2.4}$$

is the perturbation term. The parameters τ_r, $r = 1(1)m + s$, are to be determined,

$$T_r(x) = Cos\left[rCos^{-1}\left\{\frac{2x-a-b}{b-a}\right\}\right] \equiv \sum_{k=0}^{r} C_k^{(r)} x^k \tag{2.5}$$

is the Chebyshev polynomial valid in the interval $[a, b]$ (assuming that (2.1) is defined in this interval) and

$$s = max\left\{N_r - r \qquad \| 0 \leq r \leq m\right\} \qquad (2.6)$$

2.1 THE GENERALIZED CANONICAL POLYNOMIAL FOR OVERDETERMINED $m - th$ ORDER ODEs

The canonical polynomials for the initial value problems (2.1) will be obtained in this section for cases $m = 1$, 2,3 and 4 before the general formula is obtained. Since we shall be considering overdetermined cases, the formulae for $s =$ 1,2 and 3 will be presented before that of general s (s is the number of overdetermination).

Case $m = 1$, $s = 1$

$$(P_{0,0} + P_{0,1}x)\, y(x) + \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right) y'(x) = \sum_{r=0}^{F} f_r x^r, F \leq n \tag{2.7}$$

$$L \equiv \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right)\frac{d}{dx} + (P_{0,0} + P_{0,1}x)$$

$$Lx^r = \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right) rx^{r-1} + (P_{0,0} + P_{0,1}x)\, x^r$$

$$Lx^r = rP_{1,0}x^{r-1} + (rP_{1,1} + P_{0,0})\, x^r + (rP_{1,2} + P_{0,1})\, x^{r+1}$$

$$Lx^r = rP_{1,0}LQ_{r-1}(x) + (rP_{1,1} + P_{0,0})\, LQ_r(x) + (rP_{1,2} + P_{0,1})\, LQ_{r+1}(x)$$

$$Lx^r = L\left(rP_{1,0}Q_{r-1}(x) + (rP_{1,1} + P_{0,0})\, Q_r(x) + (rP_{1,2} + P_{0,1})\, Q_{r+1}(x)\right)$$

Due to the existence of L^{-1} as a result of linearity of L,

$$x^r = rP_{1,0}Q_{r-1}(x) + (rP_{1,1} + P_{0,0})\, Q_r(x) + (rP_{1,2} + P_{0,1})\, Q_{r+1}(x)$$

From where $Q_{r+1}(x)$ is obtained as

$$Q_{r+1}(x) = \frac{x^r - rP_{1,0}Q_{r+1}(x) - (rP_{1,1} + P_{0,0})\, Q_r(x)}{rP_{1,2} + P_{0,1}}, r \geq 0 \tag{2.8}$$

when $r = 0$,

$$Q_1(x) = \frac{1}{P_{0,1}} - \frac{P_{0,0}}{P_{0,1}} Q_0(x)$$

$r = 1$,

$$Q_2(x) = \frac{x}{P_{1,2} + P_{0,1}} - \frac{P_{1,1} + P_{0,0}}{P_{0,1}\left(P_{1,2} + P_{0,1}\right)} + \frac{P_{0,0}\left(P_{1,1} + P_{0,0}\right) - P_{0,1}P_{1,0}}{P_{0,1}\left(P_{1,2} + P_{0,1}\right)} Q_0(x)$$

$r = 2$,

$$Q_3(x) = \frac{x^2}{2P_{1,2} + P_{0,1}} - \frac{2P_{1,1} + P_{0,0}}{\left(P_{1,2} + P_{0,1}\right)\left(2P_{1,2} + P_{0,1}\right)} x + \frac{\left(2P_{1,1} + P_{0,0}\right)\left(P_{1,1} + P_{0,0}\right) - P_{1,0}\left(P_{1,2} + P_{0,1}\right)}{P_{0,1}\left(P_{1,2} + P_{0,1}\right)\left(2P_{1,2} + P_{0,1}\right)} - \frac{P_{0,0}\left(2P_{1,1} + P_{0,0}\right)\left(P_{1,1} + P_{0,0}\right) - P_{0,1}P_{1,0}\left(2P_{1,1} + P_{0,0}\right)}{P_{0,1}\left(P_{1,2} + P_{0,1}\right)\left(2P_{1,2} + P_{0,1}\right)} Q_1(x) + \frac{P_{1,0}P_{0,0}}{P_{0,1}\left(2P_{1,2} + P_{0,1}\right)} Q_0(x)$$

Case $m = 2,\ s = 1$

$$\left(P_{2,0} + P_{2,1}x + P_{2,2}x^2 + P_{2,3}x^3\right) y''(x) + \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right) y'(x) + \left(P_{0,0} + P_{0,1}x\right) y(x) = \sum_{F}^{r=0} f_r x^r \tag{2.9}$$

Following the same procedure as in the case $m = 1$, we have

$$Q_{r+1}(x) = \frac{x^r - r\left(r-1\right)P_{2,0}Q_{r-2}(x) - \left[r\left(r-1\right)P_{2,1} + rP_{1,0}\right]Q_{r-1}(x)}{r\left(r-1\right)P_{2,3} + rP_{1,2} + P_{0,1}} - \frac{\left[r\left(r-1\right)P_{2,2} + rP_{1,1} + P_{0,0}\right]Q_r(x)}{r\left(r-1\right)P_{2,3} + rP_{1,2} + P_{0,1}}, r \geq 0 \tag{2.10}$$

Case $m = 3,\ s = 1$

$$\left(P_{3,0} + P_{3,1}x + P_{3,2}x^2 + P_{3,3}x^3 + P_{3,4}x^4\right) y'''(x) + \left(P_{2,0} + P_{2,1}x + P_{2,2}x^2 + P_{2,3}x^3\right) y''(x) + \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right) y'(x) + \left(P_{0,0} + P_{0,1}x\right) y(x) = \sum_{F}^{r=0} f_r x^r \tag{2.11}$$

The $Q_{r+1}(x)$ for this case is obtained as

$$Q_{r+1}(x) = \frac{x^r - [r(r-1)(r-2)P_{3,0}]Q_{r-3}(x)}{P_{0,1} + rP_{1,2} + r(r-1)P_{2,3} + r(r-1)(r-2)P_{3,4}} - \frac{[r(r-1)(r-2)P_{3,1} + r(r-1)P_{2,0}]Q_{r-2}(x)}{P_{0,1} + rP_{1,2} + r(r-1)P_{2,3} + r(r-1)(r-2)P_{3,4}} - \frac{[r(r-1)(r-2)P_{3,2} + r(r-1)P_{2,1} + rP_{1,0}]Q_{r-1}(x)}{P_{0,1} + rP_{1,2} + r(r-1)P_{2,3} + r(r-1)(r-2)P_{3,4}} - \frac{[r(r-1)(r-2)P_{3,3} + r(r-1)P_{2,2} - rP_{1,1} - P_{0,0}]Q_r(x)}{P_{0,1} + rP_{1,2} + r(r-1)P_{2,3} + r(r-1)(r-2)P_{3,4}} \tag{2.12}$$

Studying the pattern of $Q_{r+1}(x)$ for $m = 1$, 2, and 3 above, we arrived at general formula for case $s = 1$ as:

$$Q_{r+1}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{m}^{j=k} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} \frac{-\sum_{j=o}^{m} j!\binom{r}{j}P_{j,j}Q_r(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}}, r \geq 0 \tag{2.13}$$

Now for $s = 2$ cases, $Q_{r+2}(x)$ we obtained:

$$Q_{r+2}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} \frac{-\sum_{k=0}^{1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j-1}\right)Q_{r+1}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}}, r \geq 0 \tag{2.14}$$

We equally obtained for case $s = 3$:

$$Q_{r+3}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+2}} \frac{-\sum_{k=0}^{2}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+2}\right)Q_{r+2}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+2}}, r \geq 0 \tag{2.15}$$

Continuing with these process, we derived for case $m = m$ and $s = s$ the general canonical polynomial

$$Q_{r+s}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} \frac{-\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}}, r \geq 0 \tag{2.16}$$

THEOREM

Let m be the order of the ODE (1.1) and let s be the number of overdeterminations, then the canonical polynomial associated with the DE is

$$Q_{r+s}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} - \frac{\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}}, r \geq 0 \qquad (2.17)$$

PROOF:

We shall employ the principles of mathematical induction over the summation variables m and s to establish the validity of (2.17).This will be achieved by varying one of these variables at a time while the other is fixed. Firstly, let s be fixed at one in (2.17) so that

$$Q_{r+1}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{m}\left(j!\binom{r}{j}P_{j,j+k}\right)Q_{r}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}}, r \geq 0 \qquad (2.18)$$

We use induction on m for fixed $s = 1$. We shall show that the formula (2.18) holds for $m = 1$:

$$Q_{r+1}(x) = \frac{x^r - \sum_{k=1}^{1}\left(\sum_{j=1}^{1} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{1} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{1}\left(j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}(x)}{\sum_{k=0}^{1} k!\binom{r}{k}P_{k,k+1}}, r \geq 0 \qquad (2.19)$$

$$Q_{r+1}(x) = \frac{x^r - rP_{1,0}Q_{r-1}(x) - (P_{0,0} + rP_{1,1})\,Q_r(x)}{P_{0,1} + rP_{1,2}} \qquad (2.20)$$

which is the same as $Q_{r+1}(x)$ in (2.13). Hence the formula (2.17) is true for $m = 1$. Now assume that (1.2) is true for $m = n$. Thus (2.17) becomes

$$Q_{r+1}(x) = \frac{x^r - \sum_{k=1}^{n}\left(\sum_{j=k}^{n} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{n} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{n}\left(j!\binom{r}{j}P_{j,j}\right)Q_{r}(x)}{\sum_{k=0}^{n} k!\binom{r}{k}P_{k,k+1}}, r \geq 0 \qquad (2.21)$$

We now show that the formula (2.17) holds for $m = n + 1$.
From our construction of $Q_{r+1}(x)$ in (2.18) for $m = 1$ up to $m = n + 1$, we have

$$Q_{r+1}(x) = \frac{x^r - \sum_{k=1}^{n}\left(\sum_{j=k}^{n} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{n} k!\binom{r}{k}P_{k,k+1} + P_{n+1,n+2}\,(n+1)!\binom{r}{n+1}} - \frac{\left(P_{n+1,n-k+1}\,(n+1)!\binom{r}{n+1}\right)Q_{r-k}(x)}{\sum_{k=0}^{n} k!\binom{r}{k}P_{k,k+1} + P_{n+1,n+2}\,(n+1)!\binom{r}{n+1}} - \frac{\sum_{j=0}^{n}\left(j!\binom{r}{j}P_{j,j}\right)Q_r(x) + P_{n+1,n+1}\,(n+1)!\binom{r}{n+1}Q_r(x)}{\sum_{k=0}^{n} k!\binom{r}{k}P_{k,k+1} + P_{n+1,n+2}\,(n+1)!\binom{r}{n+1}} \tag{2.22}$$

$$Q_{r+1}(x) = \frac{x^r - \left(\sum_{k=1}^{n}\left(\sum_{j=k}^{n} j!\binom{r}{j}P_{j,j-k}\right)\right)}{\sum_{k=0}^{n+1} k!\binom{r}{k}P_{k,k+1}} - \frac{\left(P_{n+1,n-k+1}\,(n+1)!\binom{r}{n+1}\right)Q_{r-k}(x)}{\sum_{k=0}^{n+1} k!\binom{r}{k}P_{k,k+1}} \frac{-\left(\sum_{j=0}^{n}\left(j!\binom{r}{j}P_{j,j}\right) + P_{n+1,n+1}\,(n+1)!\binom{r}{n+1}\right)Q_r(x)}{\sum_{k=0}^{n+1} k!\binom{r}{k}P_{k,k+1}} \tag{2.23}$$

$$Q_{r+1}(x) = \frac{x^r - \sum_{k=1}^{n+1}\left(\sum_{j=k}^{n+1} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{n+1} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{n+1}\left(j!\binom{r}{j}P_{j,j}\right)Q_r(x)}{\sum_{k=0}^{n+1} k!\binom{r}{k}P_{k,k+1}} \tag{2.24}$$

Thus, (2.17) holds for $m = n + 1$ and hence, from the above steps, for all positive integral values of m.
Next, we assume that (2.17) holds for $s = n$, that is

$$Q_{r+n}(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+n}} - \frac{\sum_{k=0}^{n-1}\left(\sum_{j=0}^{m}\left(j!\binom{r}{j}P_{j,j+k}\right)\right)Q_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+n}} \tag{2.25}$$

and then show that it holds for $s = n+1$, that is

$$\begin{aligned} Q_{r+n+1}(x) = & \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m}\left(k!\binom{r}{k}P_{k,k+1} + k!\binom{r}{k}P_{k,k+n+1}\right)} \\ & - \frac{\left(\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n}\left(j!\binom{r}{j}P_{j,j+k}\right)\right)Q_{r+k}(x)\right)Q_{r+n+1}(x)}{\sum_{k=0}^{m}\left(k!\binom{r}{k}P_{k,k+1} + k!\binom{r}{k}P_{k,k+n+1}\right)} \\ & - \frac{\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+n+1}\right)Q_{r+n+1}(x)}{\sum_{k=0}^{m}\left(k!\binom{r}{k}P_{k,k+1} + k!\binom{r}{k}P_{k,k+n+1}\right)} \end{aligned} \quad (2.26)$$

$$\begin{aligned} Q_{r+n+1}(x) = & \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+n+1}} \\ & - \frac{\sum_{k=0}^{n}\left(\sum_{j=0}^{m}\left(j!\binom{r}{j}P_{j,j+k}\right)\right)Q_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+n+1}} \end{aligned} \quad (2.27)$$

Thus, (2.17) holds for all m and s.

3.0 THE $n-th$ DERIVATIVES OF THE CANONICAL POLYNOMIALS OF $m-th$ ORDER OVERDETERMINED ORDINARY DIFFERENTIAL EQUATIONS ODES

The $n-th$ derivatives of the canonical polynomials presented in section 2 above are presented in this section. This is achieved by first obtaining the derivatives for individual cases and from that, now seek the general $n-th$ derivatives for all cases. As in the previous sections, m shall be the order of the differential equation, s the number of overdetermination and n, the order of the derivatives.

Case $m = 1$, $s = 1$, $n = 1$

$$Q'_{r+1}(x) = \frac{rx^{r-1} - rP_{1,0}Q'_{r-1}(x) - (P_{0,0} + rP_{1,1})\, Q'_r(x)}{P_{0,1} + rP_{1,2}}, r \geq 0 \tag{3.1}$$

Case $m = 1$, $s = 1$, $n = 2$

$$Q''_{r+1}(x) = \frac{r\,(r-1)\, x^{r-2} - rP_{1,0}Q''_{r-1}(x) - (P_{0,0} + rP_{1,1})\, Q''_r(x)}{P_{0,1} + rP_{1,2}}, r \geq 0 \tag{3.2}$$

Case $m = 1$, $s = 1$, $n = 3$

$$Q'''_{r+1}(x) = \frac{r\,(r-1)\,(r-2)\, x^{r-3} - rP_{1,0}Q'''_{r-1}(x) - (P_{0,0} + rP_{1,1})\, Q'''_r(x)}{P_{0,1} + rP_{1,2}}, r \geq 0 \tag{3.3}$$

If we continue with this process, we shall have for case $n = n$,

Case $m = 1$, $s = 1$, $n = n$

$$Q^n_{r+1}(x) = \frac{n!\binom{r}{n} x^{r-n} - rP_{1,0}Q^n_{r-1}(x) - (P_{0,0} + rP_{1,1})\, Q^n_r(x)}{P_{0,1} + rP_{1,2}}, r \geq 0 \tag{3.4}$$

Following the same procedure, we shall obtain the following results for the specific cases.

Case $m = 1$, $s = 2$, $n = n$

$$Q^n_{r+2}(x) = \frac{n!\binom{r}{n} x^{r-n} - rP_{1,0}Q^n_{r-1}(x)}{P_{0,2} + rP_{1,3}} - \frac{(P_{0,0} + rP_{1,1})\, Q^n_r(x) - (P_{0,1} + rP_{1,2})\, Q^n_{r+1}(x)}{P_{0,2} + rP_{1,3}}, r \geq 0 \tag{3.5}$$

Case $m = 1$, $s = 3$, $n = n$

$$Q^n_{r+3}(x) = \frac{n!\binom{r}{n} x^{r-n} - rP_{1,0}Q^n_{r-1}(x)}{P_{0,3} + rP_{1,4}} - \frac{(P_{0,0} + rP_{1,1})\, Q^n_r(x) - (P_{0,2} + rP_{1,3})\, Q^n_{r+1}(x)}{P_{0,3} + rP_{1,4}}, r \geq 0 \tag{3.6}$$

So that the first derivative for the case $m = m$ and $s = s$ is

$$Q'_{r+s}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q'_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} - \frac{\sum_{k=0}^{s-1}\sum_{j=0}^{m}\left(j!\binom{r}{j}P_{j,j+k}\right)Q'_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} \quad (3.7)$$

The second derivative:

$$Q''_{r+s}(x) = \frac{r(r-1)x^{r-2} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q''_{r-k}(x)}{} - \frac{\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q''_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} \quad (3.8)$$

The third derivative:

$$Q'''_{r+s}(x) = \frac{r(r-1)(r-2)x^{r-3} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q'''_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} - \frac{\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q'''_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} \quad (3.9)$$

Thus, the $n-th$ derivative is obtained as

$$Q^{(n)}_{r+s}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q^{(n)}_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} - \frac{\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q^{(n)}_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} \quad (3.10)$$

THEOREM

If the generalized canonical polynomials associated with $m-th$ order overdetermined ODE is given as

$$Q_{r+s}(x) = \frac{x^{r} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} - \frac{\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}}, r \geq 0 \quad (3.11)$$

Then its $n-th$ derivative is

$$Q_{r+s}^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} - \frac{\sum_{k=0}^{s-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}^{(n)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+s}} \tag{3.12}$$

PROOF

We shall employ here again the principle of mathematical induction over the summation variables m and s to establish the validity of (3.12). We shall vary one of these variables at a time while the other is fixed.
Firstly, let us fix s at one in (3.12) so that

$$Q_{r+1}^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{m}\left(j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}^{(n)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} \tag{3.13}$$

We use induction on m for $s=1$. We shall show that the formula holds for $m=1$;

$$Q_{r+1}^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{1}\left(\sum_{j=k}^{1} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{1} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{1}\left(j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}^{(n)}(x)}{\sum_{k=0}^{1} k!\binom{r}{k}P_{k,k+1}} \tag{3.14}$$

$$Q_{r+1}^{n}(x) = \frac{n!\binom{r}{n}x^{r-n} - rP_{1,0}Q_{r-1}^{n}(x) - (P_{0,0} + rP_{1,1})\,Q_r^n(x)}{P_{0,1} + rP_{1,2}} \tag{3.15}$$

which is the same as $Q_{r+1}(x)$ in (3.4). Hence the formula (3.13) is true for $m=1$.
Now assume that is true for $m=q$. Thus (3.13) becomes

$$Q_{r+1}^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{q}\left(\sum_{j=k}^{q} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{q}\left(j!\binom{r}{j}P_{j,j+k}\right)Q_{r+k}^{(n)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1}} \tag{3.16}$$

We now show that the formula (3.13) holds fpr $m = q + 1$.
From our construction of $Q^n_{r+1}(x)$ in (3.4) for $m = 1$ up to $m = q + 1$, we have

$$Q^{(n)}_{r+1}(x) = \frac{n!\binom{r}{n}x^{r-n} - \left(\sum_{k=1}^{q}\left(\sum_{j=k}^{q} j!\binom{r}{j}P_{j,j-k}\right)Q^n_{r-k}(x)\right)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1} + P_{q+1,q+2}\,(q+1)!\binom{r}{q+1}} + \frac{\left(\left(P_{q+1,q-k+1}\,(q+1)!\right)Q^n_{r-k}(x)\right)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1} + P_{q+1,q+2}\,(q+1)!\binom{r}{q+1}} - \frac{\sum_{j=0}^{q}\left(j!\binom{r}{j}P_{j,j}Q^n_r(x) + \left(P_{q+1,q+1}\,(q+1)!\binom{r}{q+1}\right)Q^n_r(x)\right)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1} + P_{q+1,q+2}\,(q+1)!\binom{r}{q+1}} \tag{3.17}$$

$$Q^{(n)}_{r+1}(x) = \frac{n!\binom{r}{n}x^{r-n} - \left(\sum_{k=1}^{q}\left(\sum_{j=k}^{q} j!\binom{r}{j}P_{j,j-k}\right)\right)Q^n_{r-k}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1} + P_{q+1,q+2}\,(q+1)!\binom{r}{q+1}} + \frac{\left(\left(P_{q+1,q-k+1}\,(q+1)!\right)\right)Q^n_{r-k}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1} + P_{q+1,q+2}\,(q+1)!\binom{r}{q+1}} - \frac{\left(\sum_{j=0}^{q}\left(j!\binom{r}{j}P_{j,j}Q^n_r(x) + \left(P_{q+1,q+1}\,(q+1)!\binom{r}{q+1}\right)\right)\right)Q^n_r(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k+1} + P_{q+1,q+2}\,(q+1)!\binom{r}{q+1}} \tag{3.18}$$

$$Q^{(n)}_{r+1}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{q+1}\left(\sum_{j=k}^{q+1} j!\binom{r}{j}P_{j,j-k}\right)Q^{(n)}_{r-k}(x)}{\sum_{k=0}^{q+1} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{j=0}^{q+1}\left(j!\binom{r}{j}P_{j,j}\right)Q^{(n)}_r(x)}{\sum_{k=0}^{q+1} k!\binom{r}{k}P_{k,k+1}} \tag{3.19}$$

Thus (3.12) holds for $m = q + 1$ and hence, from the steps above, for all integral values of m.
Next we assume that (3.12) holds for s=q, that is

$$Q^{(n)}_{r+q}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q^{(n)}_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} - \frac{\sum_{k=0}^{q-1}\left(\sum_{j=0}^{m}\left(j!\binom{r}{j}P_{j,j+k}\right)Q^{(n)}_{r+k}(x)\right)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+1}} \tag{3.20}$$

and the show that it holds for $s = q + 1$, that is,

$$Q^{(n)}_{r+q+1}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q^{(n)}_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+q+1}} - \frac{\sum_{k=0}^{q-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j}\right)Q^{(n)}_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+q+1}} \tag{3.21}$$

Now, by our construction of $Q^{(n)}_{r+q+1}(x)$,

$$Q^{(n)}_{r+q+1}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q^{(n)}_{r-k}(x)}{\sum_{k=0}^{m}\left(k!\binom{r}{k}P_{k,k+1} + k!\binom{r}{k}P_{k,k+q+1}\right)} - \frac{\left(\sum_{k=0}^{q-1}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q^{n}_{r+k}(x)\right)}{\sum_{k=0}^{m}\left(k!\binom{r}{k}P_{k,k+1} + k!\binom{r}{k}P_{k,k+q+1}\right)} + \frac{\left(\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+q+1}\right)Q^{(n)}_{r+q+1}(x)\right)}{\sum_{k=0}^{m}\left(k!\binom{r}{k}P_{k,k+1} + k!\binom{r}{k}P_{k,k+q+1}\right)} \tag{3.22}$$

$$Q^{(n)}_{q+r+1}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q^{(n)}_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+q+1}} - \frac{\sum_{k=0}^{q}\left(\sum_{j=0}^{m} j!\binom{r}{j}P_{j,j+k}\right)Q^{(n)}_{r+k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k+q+1}} \tag{3.23}$$

Thus, (3.12) holds for all m and s.

CONCLUSION

The derivation of a general formula for the canonical polynomials associated with m-th order overdetermined linear ODE together with its associated n-th order derivative has been presented.

The recursive nature of the formulae makes for easy determination of particular cases for which m will be specified. The fact that the determination of canonical pollynomials is independent of the boundary conditions makes it attractive in the Tau approximation problem to the solution of ODEs, and when Tau approximations of higher degrees are needed, the process of their determination does not begin from scratch.

The polynomial reported above will, in the subsequent work, be incorporated into the Tau method for purpose of generalizing the recursive formulation of the Tau method itself.

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