ISSN: 2278-0181
Vol. 1 Issue 6, August - 2012

GENERALIZED CANONICAL COSINE TRANSFORM

A. S. GUDADHE # and A.V. JOSHI *

Govt. Vidarbha Institute of Science and Humanities, Amravati. (M. S.)
* Shankarlal Khandelwal College, Akola - 444002 (M. S.)

Abstract: As generalization of the fractional Cosine transform (FRCT), the canonical cosine transform (CCT) has been used in several areas, including optical analysis and signal processing. Besides, the canonical cosine transform is also useful for radar system analysis, filter design, phase retrieval pattern recognition, and many other verities of branches of mathematics and engineering. In this paper we have proved some important results about the analyticity theorem; Inversion theorem for canonical cosine transform, Uniqueness theorem, we have also proved the Properties of Canonical Cosine Transform.

Keywords: Linear canonical transform, Fractional Fourier Transform.

Introduction: Integral transforms had provided a well establish and valuable method for solving problems in several areas of both Physics and Applied Mathematics. The roots of the method can be stressed back to the original work of Oliver Heaviside in 1890. This method proved to be of great importance, in the initial and final value problems for partial differential equations. Due to wide spread applicability of this method for partial differential equations involving distributional boundary conditions, many of the integral transforms are extended to generalized functions.

The idea of the fractional powers of Fourier operator appeared in mathematical literature as early in 1930. It has been rediscovered in quantum mechanics by Namias [9]. He had given a systematic method for the development of fractional integral transforms by means of Eigenvalues. Later on numbers of integral transforms are extended in its fractional domain. For examples Almeida [2] had studied fractional Fourier transform, Akay [1] developed fractional Mellin transform, Pei, Ding [12] studied fractional cosine and sine transforms, etc. These fractional transforms found number of applications in signal processing, image processing, quantum mechanics etc.

Recently further generalization of fractional Fourier transform known as linear canonical transform was introduced by Moshinsky [8] in 1971. Pei, Ding [16] had studied its eigen value aspect.

Linear canonical transform is a three parameter linear integral transform which has several special cases as fractional Fourier transform, Fresnel transform, Chirp transform etc. Linear canonical transform is defined as,

$$[LCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}(a/b)t^2} \cdot e^{-i(s/b)t} f(t)\, dt, \qquad \text{for } b \neq 0$$

$= \sqrt{d}\ e^{\frac{i}{2}(cd)s^2} \cdot f(d\cdot s)$, for b = 0, with ad − bc =1,

where a, b, c, and d are real parameters independent on s and t.

1 Generalized Canonical Cosine Transform:

1.1 Definition:

The Canonical Cosine Transform $f \in E^1(R^n)$ can be defined by,

{CCT $f(t)$} (s) = < f(t), K_c (t, s) > where,

$$K_C(t,s) = \sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}(a/b)t^2} \cdot \cos\left(\frac{s}{b}t\right). \qquad \text{............ (1.1.1)}$$

Hence the generalized canonical cosine transform of $f \in E^1(R^n)$ can be defined by,

$$\{CCTf(t)\}(s) = \sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}(a/b)t^2} f(t)\, dt,$$

1.2Analyticity Theorem for Canonical Cosine Transform:

If $f \in E^1(R^n)$ and its canonical cosine transform is given by,

$$[CCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\, dt,$$

then $[CCT\ f(t)](s)$ is analytic on C^n

Proof: Let $s : (s_1, s_2, \cdots\cdots\cdots s_j \cdots\cdots s_n) \in C^n$.

We first prove that

$\frac{\partial}{\partial s_j}[CCT\ f(t)](s)$ exists,

$$\frac{\partial^n}{\partial s_j^n}\{CCT\ f(t)\}(s) = < f(t), \frac{\partial^n}{\partial s_j^n} K_c(t,s) > . \qquad \text{........ (1.2.1)}$$

Where, $K_c(t,s)=\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos.\left(\frac{s}{b}t\right).$

We prove the result for $n = 1$, the general result follows by induction.

For fixed $s_j \neq 0$ choose two concentric circles C and C^1 with centre s_j and radii r and r_1 respectively, such that $0<r<r_1<|s_j|$.

Let Δs_j be a complex increment satisfying $0<|\Delta s_j|<r$.

Consider,

$$\frac{[CCT](s_j+\Delta s_j)-[CCT](s_j)}{\Delta s_j}-< f(t),\frac{\partial}{\partial s_j}\cdot K_c(t,s)> =< f(t),\quad \Psi\Delta s_j(t)>$$

Where, $\Psi\Delta s_j(t)=\frac{1}{\Delta s_j}\left[K_c\ (t,s_1,s_2,\cdots,s_j+\Delta s_j,\cdots,s_n)-K_c(t,s)-\frac{\partial}{\partial s_j}K_c\ (t,s)\right].$

For any fixed $t\in R^n$ and any fixed integer $k=(k_1,k_2,\cdots k_n)\in N_o^n$,

Since for any fixed $t\in R^n$, fixed integer k.

$D_t^k K_c(t,s)$ is analytic inside and on C', we have by Cauchy integral formula.

$$D_t^k\Psi\Delta s_j\ (t)=D_t^k\{\frac{1}{\Delta s_j}\left[\frac{1}{2\pi i}\int_{C_1}\frac{K_c(t,s)}{z-(s_j+\Delta s_j)}dz-\frac{1}{2\pi i}\int_{C_1}\frac{K_c(t,s)}{z-s_j}dz\right]-\frac{1}{2\pi i}dz\}$$

$$D_t^k\Psi\Delta s_j\ (t)=\frac{1}{2\pi i}D_t^k\int_{C_1}K_c\ (t,s)\left(\frac{1}{\Delta s_j}\left(\frac{1}{z-s_j-\Delta s_j}-\frac{1}{z-s_j}\right)-\frac{1}{(z-s_j)^2}\right)dz$$

$$D_t^k\Psi\Delta s_j\,(t)=\frac{\Delta s_j}{2\pi i}\int_{C_1}\frac{D_t^k\ K_c\ (t,s)}{(z-s-\Delta s_j)\ (z-s_j)^2}\ dz,$$

where,

$$s=(s_1\cdots s_{j-1},z,s_{j+1},\cdots s_n)$$

$$D_t^k\Psi\Delta s_j\ (t)=\frac{\Delta s_j}{2\pi i}\int_{C_1}\frac{M(t,s)}{(z-s_j-\Delta s_j)\ (z-s_j)^2}\ dz.$$

But, for all $z\in C^1$ and t restricted to a compact subset of R^n,

$M(t,s) = D_t^k\, K_c\,(t,s)$ is bounded by constant M_1.

Therefore, we have,

$$\left|D_t^k \Psi \Delta s_j\,(t)\right| \le \left|\Delta s_j\right| \frac{M_1}{(r_1 - r)\,(r_1)}$$

Thus, as $\left|\Delta s_j\right| \to 0$, $D_t^k \Psi \Delta s_j(t)$ tends to zero uniformly on the compact subset of R^n, therefore, it follows that $\Psi \Delta s_j(t)$ converges in $E(R^n)$ to zero.

Since $f \in E^1$, we conclude that [1] also tends to zero.

$$\frac{\partial}{\partial s_j}\left\{CT.f(t)\right\}(s) = < f(t), \frac{\partial}{\partial s_j} K_c(t,s) >,$$

Also tends to zero.

Therefore, $\left\{CT\ f(t)\right\}(s)$ is differentiable with respective s_j.

But this is true, for all $j = 1, 2, \ldots\ldots, n$.

Hence $\left\{CT\ f(t)\right\}(s)$ is analytic on C^n and

$$D_s^k\left\{CT\ f(t)\right\}(s) = < f(t), D_s^k\ K_c\ (t,s) >$$

i.e. $D_s^k\left\{CT\ f(t)\right\}(s) = < f(t), D_s^k\ K_c\ (t,s) >$

1.3 Inversion theorem for canonical cosine transform:

If $\left\{CT\ f(t)\right\}(s)$ canonical cosine transform of *f(t)* is given by,

$$\left\{CT\ f(t)\right\}(s) = \sqrt{\frac{1}{2\pi i b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\,dt$$

$$\text{then, } f(t) = \sqrt{\frac{2\pi i}{b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2} \int_{-\infty}^{\infty} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cos\left(\frac{s}{b}t\right) \left\{CT\ f(t)\right\}(s)\,ds$$

Proof: The canonical cosine transform of *f(t)* is given by

$$\left\{CT\ f(t)\right\}(s) = \sqrt{\frac{1}{2\pi i b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\,dt$$

$$F\ (s)=\frac{1}{\sqrt{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\int_{-\infty}^{\infty}\cdot\ e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos\left(\frac{s}{b}t\right)\ f(t)\,dt$$

Where *{CCT f(t)}(s) = F(s)*

$$\therefore F(s)\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}=\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot f(t)\cdot\cos\left(\frac{s}{b}t\right)\ dt$$

$$\therefore F(s)\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}=\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot f(t)\cdot\cos\left(\frac{s}{b}t\right)\ dt$$

$$\therefore C_1(s)=\int_{-\infty}^{\infty}g(t)\cdot\cos\left(\frac{s}{b}t\right)\ dt$$

Where $C_1(s)=F(s)\sqrt{2\pi ib}\cdot e^{-\frac{1}{2}\left(\frac{d}{b}\right)s^2}$

and $g(t)=e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot f(t)$.

$$C_1(s)=\int_{-\infty}^{\infty}g(t)\cdot\cos(\eta.t)\ dt$$

$$\because\left(\frac{s}{b}\right)=\eta \qquad d\eta=\frac{1}{b}ds$$

$$\therefore C_1(s)=\int_{-\infty}^{\infty}g(t)\cdot\cos(\eta.t)\ d\eta.$$

$\therefore$ By using inverse formula of cosine transform.

$$\therefore g(t)=\int_{-\infty}^{\infty}C_1(s)\cdot\cos(\eta.t)\ d\eta$$

$$\therefore e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot f(t)=\int_{-\infty}^{\infty}F(s)\cdot\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos(\eta.t)\ d\eta$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}F(s)\cdot\sqrt{2\pi ib}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos(\eta.t)\;d\eta.$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\sqrt{2\pi ib}\cdot F(s)\cos(\eta.t)\;\frac{1}{b}\,ds.$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{2\pi ib}\cdot\frac{1}{b}\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\;F(s)\cdot ds$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{\frac{2\pi i}{b}}\cdot\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\;F(s)\cdot ds$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{\frac{2\pi i}{b}}\cdot\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\;F(s)\cdot ds$$

$$f(t)=\sqrt{\frac{2\pi i}{b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\;\{CT\;\;f(t)\}(s)ds$$

1.4 Uniqueness Theorem:

If $\{CT\;f(t)\}(s)$ and $\{CT\;g(t)\}(s)$ are canonical cosine transform and

sup $f\subset s_\alpha,\quad s_\alpha=\{x: x\in R, |x|\le\alpha\}$ and

sup $g\subset s_\alpha,\quad s_\alpha=\{x: x\in R, |x|\le\alpha\}$ and if

$\{CT\,[f(t)]\}(s)=\{CT\;g(t)\}(s)$

then, $f=g$ in the sense of equality in $D'(I)$.

Proof: By inversion theorem

$$\therefore\; f-g=\left\{\sqrt{\frac{2\pi i}{b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\;\{CT\;f(t)\}(s)ds\right\}$$
$$-\left\{\sqrt{\frac{2\pi i}{b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\;\{CT\;g(t)\}(s)ds\right\}$$

$\therefore\ f-g=\sqrt{\frac{2\pi i}{b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot\int_{-\infty}^{\infty} e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot\cos\left(\frac{s}{b}t\right)\left[\{CCT\ f(t)\}(s)-\{CCT\ g(t)\}(s)\right]ds$ Thus f = g in $D'(I)$.

1.5 Properties of Canonical Cosine Transform:

1. 5. 1 Shifting property of canonical cosine transform:

If *{CCT f(t)} (s)* denotes generalized canonical cosine transform of *f(t)* and 'τ, is any real number. Then,

$$\{CCT\ f(t+\tau)\}(s)=e^{\frac{i}{2}\left(\frac{a}{b}\right)\tau^2}\ [\cos\left(\frac{s}{b}\tau\right)\ \left\{CCT\ f(t)\cdot e^{-it\tau\left(\frac{a}{b}\right)}\right\}(s)$$

$$+\sin\left(\frac{s}{b}\tau\right)\ \left\{CST\ f(t)\cdot e^{-it\tau\left(\frac{a}{b}\right)}\right\}(s)]$$

1. 5. 2 Differentiation property of canonical cosine Transform:

If *{CCT f(t)}(s)* denotes generalized canonical cosine transform of *f(t),* then

$$\{CCT\ (f^1(t))\}(s)=\left(\frac{s}{b}\right)\{CST\ f(t)\}(s)-i\left(\frac{a}{b}\right)\ \{CCT\ f(t)\}(s)$$

1. 5. 3 Scaling property of canonical cosine transform:

If *{*CCT *f(t)}(s)* denotes generalized canonical cosine transform, then

$$\{CCT\,[f(kt)]\}(s)=\frac{1}{k}\,e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\left(\frac{d}{b}\right)s^2}\left[CCT\left\{f(t)\cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\left(\frac{a}{bk}\right)t^2}\right\}\right](s)$$

Conclusion: In this paper, brief introduction of the generalized canonical cosine transform is given and its analyticity theorem, Inversion theorem for canonical cosine transform, Uniqueness theorem is proved. Properties of Canonical Cosine Transform are also obtained which will be useful in solving differential equations occurring in signal processing and many other branches of engineering.

Acknowledgement: The author is thankful to referee, for his valuable comments. The suggestions made by Professor A. S. Gudadhe have been very helpful in this investigation.

References:

[1] Akay O. and Bertels, (1998): Fractional Mellin Transformation: An extension of fractional frequency concept for scale, 8th IEEE, Dig. Sign. Proc. Workshop, Bryce Canyan, Utah.

[2] Almeida, L.B., (1994): The fractional Fourier Transform and time-frequency representations, IEEE. Trans. on Sign. Proc., Vol. 42, No.11, 3084-3091.

[3] Bhosale B. N., and Chaudhary M. S., (2002): Fractional Fourier transform of distribution of compact support, Bull. Cal. Math. Soc., Vol.94, No.5, 349-358

[4] Gelfand I. M. and Shilov G. E., (1964): Generalized functions, Vol. I, Academic Press, New York.

[5] Gelfand I. M. and Shilov G. E., (1967): Generalized functions, Vol. I, Academic Press, New York.

[6] Lohamann, A. W.: Image Rotation, Winger Rotation and The Fractional Fourier Transform, Jour. Opt. Soc. Am. A; Vol. 10, No.10, Oct. 1993, 2181-2186.

[7] Mahalle, V.N.and Gudadhe A S: On Generalized Fractional Complex Mellin Transform AMSA, Vol. 19, No. 1,(2009) P. 31-38.

[8] Moshinsky, M.: Linear canonical transform and their unitary representation, Jour. Math, Phy.,Vol.12, No. 8 , P. 1772-1783, (1971).

[9] Namias V. (1980): The fractional order Fourier transform and its applications to quantum mechanics, Jour. Inst. Math's. App., Vol. 25, 241-265.

[10] Ozaktas H.M., Zalevsky Z., Kutay M.A., (2000): The fractional Fourier transform with applications in optics and signal processing, Pub. John .Wiley and Sons Ltd.

[11] Pathak R.S., (1997): Integral transforms of generalized functions and their applications, Gardon and Breach Science Publisher.

[12] Pie and Ding, (2001): Relations between fractional operations and time-frequency distributions and their application, IEEE. Trans. on Sign. Proc., Vol. 49, No.8, 1638-1654.

[13] Sontakke, P. K. and Gudadhe, A.S.: Analyticity and Operation Transform On Fractional Hartley Transform Int. Journal of Math. Analysis, Vol.2, 2008,

No. 20, 977-986**.**

[14] Torre, A.: Linear and Radial Canonical Transforms of Fractional Order, Jour. of Computational and applied Mathematics 153 (2003), 477 - 486.

[15] Zemanian A. H., (1968): Generalized integral transform, Inter Science Publishers, New York.

[16] Soo-Chang Pei, and Jian-Jiun Ding: Eigenfunctions of Linear Canonical Transform Vol. 50, No. 1, January (2002).