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Introducing Network Structure on Fuzzy Banach Manifold

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Abstract: **we define concept of connectedness on fuzzy Banach manifold with well defined fuzzy norm. Also we show that connectedness is an equivalence relation and induces a network structure on fuzzy Banach manifold.**

Keywords: Fuzzy Banach manifold, local path connectedness, internal path connectedness, maximal path connectedness.

1. INTRODUCTION

In our previous paper [4], we have introduced the concept of fuzzy Banach manifold and proved some related results. In this paper we define norm on fuzzy Banach manifold and hence fuzzy Banach space.

Connectedness on manifold plays very important role in geometry. S. C. P. Halakatti and H. G. Haloli[5] have introduced different forms of connectedness and network structure on differentiable manifold, using their approach we extend different forms of connectedness on fuzzy Banach manifold and show that fuzzy Banach manifold admits network structure by admitting maximal connectedness as an equivalence relation, further it will be shown that network structure on fuzzy Banach manifolds is preserved under smooth map.

2. PRELIMINARIES

Definition 2.1: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if ([0, 1], $*$) is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$

Definition 2.2: The triplets $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$;

i) $M(x, y, 0) > 0$
ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$
iii) $M(x, y, t) = M(y, x, t)$
iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$
v) $M(x, y, \cdot): (0, \infty) \to [0, 1]$ is continuous.

Definition 2.3: The triplet $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying following conditions for every $x, y \in X$ and $t, s > 0$;

i) $N(x, y) > 0$,
ii) $N(x, t) = 1$ *iff* $x = 0$,
iii) $N(\alpha x, t) = \left(x, \frac{t}{|\alpha|}\right)$ for $\alpha \neq 0$,
iv) $N(x, t) * N(y, s) \leq N(x + y, t + s)$,
v) $N(x, .): (0, \infty) \to [0, 1]$ is continuous,
vi) $lim_{t \to \infty} N(x, t) = 1$.

Lemma 2.1: Let $(X, N, *)$ is a fuzzy normed linear space. If we define $M(x, y, t) = N(x - y, t)$ then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

Definition 2.4: Let M be any fuzzy topological space, U is a fuzzy subset of M such that sup $\{\mu_U(x)\} = 1$, $\forall\, x \in M$ and φ is a fuzzy homeomorphism defined on the support of $U = \{x \in M : \mu_U(x) > 0\}$, which maps U onto an open fuzzy subset $\varphi(U)$ in some fuzzy Banach space E_i. Then the pair (U, φ) is called as fuzzy Banach chart.

Definition 2.5: A fuzzy Banach atlas A of class C^k on M is a collection of pairs (U_i, φ_i) (i $\in I$) subjected to the following conditions:

i) $\bigcup_{i \in I} U_i = M$ that is the domain of fuzzy Banach charts in A cover M.
ii) Each fuzzy homeomorphism φ_i, defined on the support of $U_i = \{x \in M : \mu_U(x) > 0\}$ which maps U_i onto an open fuzzy subset $\varphi_i(U_i)$ in some fuzzy Banach space E_i, and $\forall\, i, j \in I, \varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ are open fuzzy subsets in E_i.
iii) The maps $\varphi_i \circ \varphi_j^{-1}$ which maps $\varphi_j(U_i \cap U_j)$ onto $\varphi_i(U_i \cap U_j)$ is a fuzzy diffeomorphism of class C^k $(k \geq 1)$ for each pair of indices i, j.

The maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ for $i, j \in I$ are called fuzzy transition maps.

Definition 2.6: A fuzzy topological space M modelled on fuzzy Banach space E is called fuzzy Banach manifold.

3. NORM ON FUZZY BANACH MANIFOLD

In this section we define norm on fuzzy Banach manifold and hence fuzzy Banach space.

Definition 3.1: Let M be a fuzzy Banach manifold, then addition on M is defined as follows:

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for any $x \in U_i$, $y \in U_j$; $x+y \in A^k(M)(i,j \in I)$ ($\because M$ is a fuzzy Banach manifold).

Definition 3.2: Let M be a fuzzy Banach manifold, then scalar multiplication on M is defined as follows:
for any $x \in U_i$ $(i \in I)$ and $\alpha \in F$where F if either field of real number or complex numbers) $\alpha x \in A^k(M)$.

Definition 3.3: A fuzzy Banach manifold M on which addition and scalar multiplication is well defined forms a vector space.

Definition 3.4: Let M be a fuzzy Banch manifold which is a vector space and $*$ is a continuous t-norm and N is a fuzzy set on $M \times (0,\infty)$ satisfying the following conditions for any $x \in U_i, y \in U_j$ $(i,j \in I)$, t, $s > 0$, and $\alpha \in F$.

i) $\forall\, t \in \mathbb{R}$ with $t \leq 0, N(x,t) = 0$
ii) $\forall\, t \in \mathbb{R}, t > 0: N(x,t) = 1$ iff $x = 0$;
iii) $\forall\, t \in \mathbb{R}, t > 0$:
  i) $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$ if $\alpha \neq 0$.
  ii) $N(c_1 x + c_2 y, t+s) = N\left(x+y, \frac{t+s}{c_1+c_2}\right)$ if $c_1 + c_2 \neq 0$.
iv) $\lim_{n\to\infty} N(x,t) = 1$.

Then the triplet $(M,N,*)$ is called as a fuzzy normed linear fuzzy Banach manifold.

Definition 3.5: Let $(M,N,*)$ be a fuzzy Banach manifold and is a fuzzy normed linear space. Let $\{x_n\}$ be a sequence of elements in$(M,N,*)$. Then $\{x_n\}$ is said to be convergent if $\exists\, x \in U_i$ (i $\in I$) such that:

$$\lim_{n\to\infty} N(x_n - x, t) = 1, \forall\, t > 0.$$

Then x is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty}\{x_n\}$.

Definition 3.6: A sequence $\{x_n\}$ in $(M,N,*)$ is said to be a Cauchy sequence if

$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1, \forall\, t > 0 \text{ and } p = 1,2,3 \ldots.$$

Definition 3.7: A fuzzy Banach manifold $(M,N,*)$ which is a fuzzy normed linear space is said to be complete if every Cauchy's sequence in $(M,N,*)$ converges in $(M,N,*)$. Then the triplet $(M,N,*)$ is called a fuzzy Banach space.

4. CONNECTEDNESS ON FUZZY BANACH MANIFOLD

Let $p,q,r,s \in M$, by the property of fuzzy metric defined by fuzzy norm given by $M(p,q,t) * M(q,r,s) \leq M(p,r,t+s)$ intuitively means, there is a triangular relation between elements of M. This relation leads to define local, internal and maximal connectivity on fuzzy Banach manifold.

Definition 4.1: Let $(M,N,*)$ be a fuzzy Banach manifold and $p,q \in U_i \in A^k(M)$ if there exists a path $\gamma: [0,1] \to U_i$ such that:

$$\gamma(0) = p \in U_i$$

$$\gamma(1) = q \in U_i$$

Then p is locally path connected to q. If it is true for all $p,q \in M$ then M is locally path connected.

Definition 4.2: Let $(M,N,*)$ be a fuzzy Banach manifold and $p \in U_i$, $q \in U_j$ if there exists a path $\gamma: [0,1] \to U_i \cup U_j$ such that:

$$\gamma(0) = p \in U_i$$

$$\gamma(1) = q \in U_j$$

Then p is internally path connected to q. If it is true for all $p \in U_i$, $q \in U_j$ (i, j$\in$ I) respectively then M is internally path connected.

Definition 4.3: Let $(M,N,*)$ be a fuzzy Banach manifold and $p \in U_i, q \in U_j, r \in U_k$ if there exists a path $\gamma: [0,1] \to A^k(M)$ such that:

$$\gamma(0) = p \in U_i$$

$$\gamma\left(\tfrac{1}{2}\right) = q \in U_j$$

$$\gamma(1) = r \in U_k$$

Then p is maximally path connected to r. If it is true for all $p,q,r \in U_i, U_j, U_k$ $\forall$ i, j, k $\in$ I respectively then M is maximally path connected.

Now, we shall show that maximal path connectedness on M is an equivalence relation.

Theorem 4.1: A maximally path connectedness on fuzzy Banach manifold $(M,N,*)$ is an equivalence relation.

Proof: Let $(M,N,*)$ be a fuzzy Banach manifold. Now we shall show that maximal path connectedness on $(M,N,*)$ is an equivalence relation.

We say p is maximally path connected to r if for $p \in U_i, q \in U_j, r \in U_k$ there exists a path $\gamma: [0,1] \to A^k(M)$ such that:

$$\gamma(0) = p \in U_i$$

$$\gamma\left(\tfrac{1}{2}\right) = q \in U_j$$

$$\gamma(1) = r \in U_k$$

1) Reflexive: Reflexive relation is trivial by considering a constant paths i.e.,

$$\gamma: [0,1] \to A^k(M)$$

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$$\gamma(t) = p, \forall t \in [0,1]$$

2) Symmetry: Suppose γ_1 is a path from p to r then let

$$\gamma_2: [0,1] \to A^k(M)$$

such that:

$$\gamma_2(t) = \gamma_1(1-t), \forall t \in [0,1]$$

γ_2 is continuous since it is the composition of $\gamma_1 \circ f$ where $f: [0,1] \to [0,1]$ is the map $f(t) = (1-t), t \in [0,1]$.

Then γ_2 is a path from r to p such that:

$$\gamma_2(0) = \gamma_1(1-0) = \gamma_1(1) = r$$

$$\gamma_2\left(\tfrac{1}{2}\right) = \gamma_1\left(1-\tfrac{1}{2}\right) = \gamma_1\left(\tfrac{1}{2}\right) = q$$

$$\gamma_2(1) = \gamma_1(1-1) = \gamma_1(0) = p$$

Therefore, maximal path connectedness relation is symmetric.

3) Transitivity: Suppose γ_1 is a path from p to r and γ_2 is a path from r to s.

Let $\gamma_3: [0,1] \to A^k(M)$ defined as,

$$\gamma_3(t) = \begin{cases} \gamma_1(2t) & if \quad 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & if \quad \frac{1}{2} \le t \le 1 \end{cases}$$

Then γ_3 is well defined since $\gamma_1(1) = r = \gamma_2(0)$. Also γ_3 is continuous since its restriction to each of the two closed subsets $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ of $[0,1]$ is continuous.

i.e.,

$$\gamma_3(0) = \gamma_1(0) = p$$

$$\gamma_3\left(\tfrac{1}{2}\right) = \begin{cases} \gamma_1\left(2\left(\tfrac{1}{2}\right)\right) = \gamma_1(1) = r \\ \gamma_2\left(2\left(\tfrac{1}{2}\right) - 1\right) = \gamma_2(0) = r \end{cases}$$

$$\gamma_3(1) = \gamma_2(2(1)-1) = \gamma_2(1) = s$$

Therefore, γ_3 is a path from p to s.

Hence maximal path connectedness relation is transitive.

Therefore, maximal path connectedness is an equivalence relation.

The equivalence relation maximal path connectedness on $(M, N, *)$ induces a network structure.

Definition 4.4: A fuzzy Banach manifold $(M, N, *)$ admitting maximal path connectedness relation as an equivalence relation induces a network structure on $(M, N, *)$ denoted by $(M, N, *, \sim)$.

Now we shall show that any two fuzzy Banach manifolds preserve network structure under smooth map.

Theorem 4.2: Let $(M_1, N_1, *)$ be a fuzzy Banach manifold admitting network structure and $(M_2, N_2, *)$ be any fuzzy Banach manifold and if $f: M_1 \to M_2$ is a smooth map then f preserves a maximal path connectedness and hence network structure on $(M_2, N_2, *)$.

Proof: Let $(M_1, N_1, *)$ be a Banach manifold admitting network structure and $(M_2, N_2, *)$ be any fuzzy Banach manifold and if $f: M_1 \to M_2$ is a smooth map.

Since $(M_1, N_1, *)$ is a Banach manifold admitting network structure, for $p \in U_i, q \in U_j, r \in U_k$ there exists a path $\gamma: [0,1] \to A^k(M)$ such that:

$$\gamma(0) = p \in U_i$$

$$\gamma\left(\tfrac{1}{2}\right) = q \in U_j$$

$$\gamma(1) = r \in U_k$$

Also, we know that f and γ are both continuous and differentiable so the composition $f \circ \gamma$ is also continuous and differentiable so we can say that for $p \in U_i q \in U_j, r \in U_k$ we have $f(p) \in f(U_i), f(q) \in f(U_j)$ and $f(r) \in f(U_k)$ where $f(U_i), f(U_j), f(U_k) \in A^k(M_2)$ such that for every path $\gamma: [0,1] \to A^k(M)$ there is a path $f \circ \gamma : [0,1] \to A^k(M_2)$ such that

$$f \circ \gamma\,(0) = f(p) \in f(U_i)$$

$$f \circ \gamma\left(\tfrac{1}{2}\right) = f(q) \in f(U_j)$$

$$f \circ \gamma\,(1) = f(r) \in f(U_k)$$

Since f is a smooth map this is true for all $p, q, r \in M_1$ and $f(p), f(q), f(r) \in M_2$.

Therefore, M_2 is maximally path connected.

Similarly it can be easily shown with the help of Theorem 4.1 that maximal connectedness on M_2 is an equivalence relation hence forms a network structure on M_2.

Example 1:

Let C be a set of all continuous functions on a closed interval [0, 1] be a Banach space with the norm $\|f\| = Sup_{x\in[0,1]} |f(x)|$, is a fuzzy Banach space $(C, \|.\|)$ and hence a fuzzy Banach manifold.

Solution: We know that $(C, \|.\|)$ is a fuzzy Banach space with the norm function given by $\|f\| = Sup_{x\in[0,1]} |f(x)|$

Let $(C, \|.\|)$ be linear space. We define $*: [0,1] \times [0,1] \to [0,1]$ as $a * b = ab$ and $N(f,t) = \frac{t}{t+\|f\|}$ where $\|f\| = Sup_{x\in[0,1]} |f(x)|$ clearly, $N(f,t)$ is a fuzzy norm defined

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by the norm $\|f\|$. Hence $(C,\ N,\ *)$ is a fuzzy normed linear space.

Now, we shall show that every Cauchy sequence in $(C,\ N,\ *)$ converges in $(C,\ N,\ *)$. For this we need to show that $\lim_{n\to\infty} N(f_{n+p} - f_n, t) = 1$

Consider,

$$\lim_{n\to\infty} N(f_{n+p} - f_n, t)$$

$$= \lim_{n\to\infty} \frac{t}{t + \|f_{n+p} - f_n\|}$$

Since $(C, \|.\|)$ is a Banach space, we have

$\lim_{n\to\infty}\|f_{n+p} - f_n\| = 0$

Therefore $\lim_{n\to\infty} \frac{t}{t+0} = 1$

i.e., $\lim_{n\to\infty} N(f_{n+p} - f_n, t) = 1$

Hence, every Cauchy sequence in $(C,\ N,\ *)$ converges in $(C,\ N,\ *)$.

Therefore, $(C,\ N,\ *)$ is a fuzzy Banach space.

Let M and E be fuzzy Banach space $(C,\ N,\ *)$. Introduce a chart $U = M = (C,\ N,\ *)$ and φ be identity mapping. This single chart satisfies all the conditions of charts and atlas hence, $M = (C,\ N,\ *)$ with identity mapping is a fuzzy Banach manifold. ■

Example 2:

Let $C(M) = \{f: U \subset M \to \mathcal{R} : U\ is\ closed\ subset\ of\ M\}$ be collection of all continuous functions defined on fuzzy Banach manifold M be a Banach space $(C, \|.\|)$ with norm $\|f\| = Sup_{x\in U}|f(x)|$, is fuzzy Banach space and hence a fuzzy Banach manifold.

Solution: We know that $C(M)$ with norm $\|f\| = Sup_{x\in U}|f(x)|$ forms a Banach space $(C, \|.\|)$. First we shall show that $(C(M), \|.\|)$ is a fuzzy Banach space.

Let $*: [0,1] \times [0,1] \to [0,1]$ be defined as $a * b = ab$ and $N(f,t) = \left(\frac{t}{t+\|f\|}\right)$ where $\|f\| = Sup_{x\in U}|f(x)|$ clearly, $N(f,t)$ is a fuzzy norm defined by the norm $\|f\|$, hence $(C(M),\ N,\ *)$ is a fuzzy normed linear space, from above example, we can say that $(C(M), N,\ *)$ is a fuzzy Banach space and with identity mapping φ, $(C(M),\ N,\ *)$ is a fuzzy Banach manifold. ■

CONCLUSION

An equivalence relation maximal connectedness on fuzzy Banach manifold induces a network structure on fuzzy Banach manifold.

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