Non Commutative Fourier Transform on Some Lie Groups and Its Application to Harmonic Analysis.

Kahar El-Hussein
Department of Mathematics, Faculty of Science,
Al Furat University, Dear El Zore, Syria and
Department of Mathematics, Faculty of Science,
Al-Jouf University, KSA

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Abstract

This paper will focus on Fourier transform on the semidirect product of two Lie groups to obtain some results in abstract harmonic analysis. In fact the combining of the classical Fourier transform on $\mathbb{R}$, and on a compact Lie group permits us to define the Fourier transform, and then to obtain the Plancherel formula on these groups. In the end we will introduce some interesting new groups.

Keywords: **Key words :** Semidirect Product of Two Lie Groups, Fourier Transform, Plancherel Formula

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1 Introduction.

1.1. Noncommutative Fourier analysis is a beautiful and powerful area of pure mathematics that has connections to, theoretical physics, chemistry analysis, algebra, geometry, and the theory of algorithms. In mathematics, abstract harmonic analysis is the field in which results from Fourier analysis are extended to topological groups which are not commutative. For a long time, people have tried to construct objects in order to generalize Fourier transform and Pontryagin,s theorem to the non abelian case. However, with the dual object not being a group, it is not possible to define the Fourier transform and the inverse Fourier transform between G and $\widehat{G}$. These difficulties of Fourier analysis on noncommutative groups makes the noncommutative version of the problem very challenging. It was necessary to find a subgroup or at least a subset of locally compact groups which were not "pathological", or "wild" as Kirillov calls them [13]. Unfortunatly If the group G is no longer assumed to be abelian, it is not possible anymore to consider the dual group $\widehat{G}$ (i.e the set of all equivalence classes of unitary irreducible representations). Abstract harmonic analysis on locally compact groups is generally a difficult task. Still now neither the theory of quantum groups nor the representations theory have done to reach this goal. So the important and interesting question is: One can do abstract harmonic analysis on Lie groups *i.e.* the Fourier transform can be defined to solve the major problems of abstract harmonic analysis. Here are some interesting examples of these groupse.

1.2. The linear group $GL(n, \mathbb{R})$ consisting of all matrices of the form

$$GL(n, \mathbb{R}) = \{ \left(\ a_{ij} \ \right) : a_{ij} \in \mathbb{R}, \ 1 \leq i \leq n, \ 1 \leq j \leq n \} \tag{1}$$

The orthogonal group $O(n, \mathbb{R})$

$$O(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) : AA^{\star} = I \ and \ \det A = 1 \ \} \tag{2}$$

1.3. The general Lorentz groupThe Lorentz group provides another interesting example. Moreover, the Lorentz group $O(3, 1)$ shows up in an interesting way in computer vision. Denote the $p \times p-$identity matrix by $I_{p,p}$ and define

$$I_{p,q} = \begin{pmatrix} I_{p,p} & 0 \\ 0 & -I_{q,q} \end{pmatrix}$$

We denote by $O(p, q)$ the group consisting of all matrices of the form

$$O(p, \ q) = \{ A \in GL(n, \mathbb{R}), \ A^t I_{p,q} A = I_{p,q} \ \} \tag{3}$$

1.4. The $n-$dimensional real Heisenberg group H. the Galelian group GA which is isomorphic onto the group $H\rtimes SO(3, \mathbb{R})$ semidirect product of H by $SO(3, \mathbb{R})$, *i.e* $GA \simeq H\rtimes SO(3, \mathbb{R})$. Let $SL(2, \mathbb{R})$ be the 2×2 real semisimple Lie group and let $SL(2, \mathbb{C})$ be the 2×2 complex semisimple Lie group, then we get the Jacobi group $G^J \simeq H\rtimes SL(2, \mathbb{R})$ and the Poincare group(Space time) $\simeq \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$.

Recently, these problems found a satisfactory solution with the papers $[6, 8, 10, 11]$. The ways were introduced in those papers will be the business of the expertise in the theory of abstract harmonic analysis, and in theoretical physics, and that is what I am interested In this paper I will define the Fourier transform and establishing Plancherel formula for the semidirect of two vector groups $\mathbb{R}^n \rtimes \mathbb{R}^m$ $(m \leq n)$ and the motion group.

2 Fourier Transform and Plancherel Formula for the Semidirect Product Lie groups.

2.1. Let $L = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ be the group with law:

$$(x, t, r)(y, s, q) = (x + \rho(r)y, t + s, r + q)$$

for all $(x, t, r) \in L$ and $(y, s, q) \in L$. In this case the group G can be identified with the closed subgroup $\mathbb{R}^n \times \{0\} \times_\rho \mathbb{R}^m$ of L and B with the subgroup $\mathbb{R}^n \times \mathbb{R}^m \times \{0\}$of L.

Definition 2.1. *For every $f \in C^\infty(G)$, one can define a function $\widetilde{f} \in C^\infty(L)$ as follows:*

$$\widetilde{f}(x, t, r) = f(\rho(t)x, r + t) \tag{4}$$

for all $(x, t, r) \in L$. So every function $\psi(x, r)$ on G extends uniquely as an invariant function $\widetilde{\psi}(x, t, r)$ on L.

Remark 2.1. *The function $\widetilde{f}$ is invariant in the following sense:*

$$\widetilde{f}(\rho(s)x, t - s, r + s) = \widetilde{f}(x, t, r) \tag{5}$$

for any $(x, t, r) \in L$ and $s \in \mathbb{R}^m$.

Lemma 2.1. *For every function $F \in C^\infty(L)$ invariant in sense (5) and for every $u \in \mathcal{U}$, we have*

$$u \ * F(x, t, r) = u \ *_c F(x, t, r) \tag{6}$$

for every $(x,t,r) \in L$, where $*$ signifies the convolution product on G with respect the variables (x,r) and $*_c$signifies the commutative convolution product on B with respect the variables (x,t).

Proof: In fact we have

$$P_u F(x,t,r) = u * F(x,t,r) = \int\limits_G F(y,s)^{-1}(x,t,r)u(y,s)dyds$$

$$= \int\limits_G F\left[(\rho(-s)(-y),-s)(x,t,r)\right]u(y,s)dyds$$

$$= \int\limits_G F\left[\rho(-s)(x-y),t,r-s\right]u(y,s)dyds$$

$$= \int\limits_G F\left[x-y,t-s,r\right]u(y,s)dyds = u *_c F(x,t,r) = Q_u F(x,t,r) \quad (7)$$

where P_u and Q_u are the invariant differential operators on G and B respectivel. As in [9], we will define the Fourier transform on G. Therefor let $\mathcal{S}(G)$ be the Schwartz space of G which can be considered as the Schwartz space of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, and let $\mathcal{S}'(G)$ be the space of all tempered distributions on G. The action ρ of the group $\mathbb{R}^m$ on $\mathbb{R}^n$ defines a natural action ρ of the dual group $(\mathbb{R}^m)^*$of the group $\mathbb{R}^m$ $((\mathbb{R}^m)^* \simeq \mathbb{R}^m)$ on $(\mathbb{R}^n)^*$, which is given by :

$$\langle \rho(t)(\xi), x\rangle = \langle \xi, \rho(t)(x)\rangle \quad (8)$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Also we have, for every $u \in \mathcal{S}(G)$ and $f \in \mathcal{S}(G)$

$$u * \widetilde{f}(x,t,r) = u *_c \widetilde{f}(x,t,r) \quad (9)$$

Definition 2.1. *If $f \in \mathcal{S}(G)$, one can define its Fourier transform $\mathcal{F}f$ by :*

$$\mathcal{F}f\ (\xi,\lambda) = \int\limits_G f(x,t)\ e^{-i\ (\langle \xi,x\rangle + \langle \lambda,t\rangle)}\ dxdt \quad (10)$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$ and *$t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$, where $\langle \xi, x\rangle = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ and*

$\langle\lambda,t\rangle = \lambda_1 t_1 + \lambda_2 t_2 + ... + \lambda_m t_m$. *It is clear that* $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^{n+m})$ *and the mapping* $f \to \mathcal{F}f$ *is isomorphism of the topological vector space* $\mathcal{S}(G)$ *onto* $\mathcal{S}(\mathbb{R}^{n+m})$.

Definition 2.2. *If* $f \in \mathcal{S}(G)$, *we define the Fourier transform of its invariant* $\widetilde{f}$ *as follows*

$$\mathcal{F}(\widetilde{f})(\xi,\lambda,0) = \int\limits_{L\times\mathbb{R}^m} \widetilde{f}(x,t,s)e^{-\ i\ (\langle\xi,x\rangle+\langle\lambda,t\rangle)}\ e^{-\ i\ \langle\mu,s\rangle}\ dxdtdsd\mu \tag{11}$$

where $(\mu,s) \in \mathbb{R}^{n+m}$ *and* $\langle\mu,s\rangle = \mu_1 s_1 + \mu_2 s_2 + ... + \mu_m s_m$

Corollary 2.1. *For every* $u \in \mathcal{S}(G)$, *and* $f \in \mathcal{S}(G)$, *we have*

$$\begin{aligned}\int\limits_{\mathbb{R}^m} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,\lambda,\mu)d\mu &= \int\limits_{\mathbb{R}^m} \mathcal{F}(\widetilde{f})(\xi,\lambda,\mu)\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)d\mu \\ &= \mathcal{F}(\widetilde{f})(\xi,\lambda,0)\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)\end{aligned} \tag{12}$$

for any $\xi = (\xi_1,\xi_2,...,\xi_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1,\lambda_2,...,\lambda_m) \in \mathbb{R}^m$ *and* $\mu = (\mu_1,\mu_2,...,\mu_m) \in \mathbb{R}^m$, *where* $\overset{\vee}{u}(x,t) = u(x,t)^{-1}$

Proof: By equation (9)we have

$$\overset{\vee}{u} * \widetilde{f}(x,t,r) == \overset{\vee}{u} *_c \widetilde{f}(x,t,r) \tag{13}$$

Applying the Fourier transom we get

$$\begin{aligned}\int\limits_{\mathbb{R}^m} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,\lambda,\mu)d\mu &= \mathcal{F}(\overset{\vee}{u} *_c \widetilde{f})(\xi,\lambda,0) \\ & \mathcal{F}(\widetilde{f})(\xi,\lambda,0)\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)\end{aligned} \tag{14}$$

Theorem 2.1.(*Plancherel's formula*). *For any* $f \in L^1(G)\cap L^2(G)$, *we get*

$$\int\limits_G |f(x,t)|^2\, dxdt = \int\limits_{\mathbb{R}^{n+m}} |\mathcal{F}f(\xi,\lambda)|^2\, d\xi d\lambda \tag{15}$$

Proof: First, let $\overset{\widetilde{\vee}}{f}$ be the function defined by

$$\overset{\widetilde{\vee}}{f}(x,t,r) = \overline{f((\rho(t)x,r+t)^{-1})} \tag{16}$$

then we have

$$f \ast \widetilde{\overset{\vee}{f}}(0,0,0) = \int_G \widetilde{\overset{\vee}{f}}\left[(x,t)^{-1}(0,0,0)\right] f(x,t)dxdt$$

$$= \int_G \widetilde{\overset{\vee}{f}}\left[\rho(-t)((-x)+(0)),0,0-t\right] f(x,t)dxdt$$

$$= \int_G \widetilde{\overset{\vee}{f}}\left[\rho(-t)(-x),0,-t\right] f(x,t)dxdt$$

$$= \int_G \overset{\vee}{f}\left[\rho(-t)(-x),-t\right] f(x,t)dxdt = \int_G \overline{f(x,t)} f(x,t)dxdt$$

$$= \int_G |f(x,t)|^2 \, dxdt \tag{17}$$

Second by (12), we obtain

$$f \ast \widetilde{\overset{\vee}{f}}(0,0,0)$$

$$= \int_{\mathbb{R}^{n+2m}} \mathcal{F}(f \ast \widetilde{\overset{\vee}{f}})(\xi,\lambda,\mu)d\xi d\lambda d\mu = \int_{\mathbb{R}^{n+2m}} \mathcal{F}(f \ast_c \widetilde{\overset{\vee}{f}})(\xi,\lambda,\mu)d\xi d\lambda d\mu$$

$$= \int_{\mathbb{R}^{n+m}} \mathcal{F}(\widetilde{\overset{\vee}{f}})(\xi,\lambda,0)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda = \int_{\mathbb{R}^{n+m}} \overline{\mathcal{F}(f)}\,(\xi,\lambda)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda$$

$$= \int_{\mathbb{R}^{n+m}} |\mathcal{F}(f)(\xi,\lambda)|^2 \, d\xi d\lambda = \int_G |f(x,t)|^2 \, dxdt \tag{18}$$

which is the Plancherel's formula on G. So the Fourier transform can be extended to an isometry of $L^2(G)$ onto $L^2(\mathbb{R}^{n+m})$.

Corollary 2.2. *In equation* (18), replace *the second* f by g, *we obtain*

$$\int_G \overline{f(x,t)} g(x,t)dxdt = \int_{\mathbb{R}^{n+m}} \overline{\mathcal{F}(f)(\xi,\lambda} \mathcal{F}g(\xi,\lambda)d\xi d\lambda \tag{19}$$

which is the Parseval formula on G.

3 Fourie Transform and Plancherel Formula for the Motion Group.

3.1. Let V be the $n-$ dimensional vectoriel group, K a compact Lie group and $\rho : K \to GL(V)$ a continuous linear representation from K in V . Let $G = V \rtimes_\rho K$ be the motion group, which is the semi-direct product of the group V and K. We supply V by$K-$invariant scalar product which is denoted by $(\,|\,)$. Let $\mathcal{S}(V)$ be the Schwartz space of V . We denote $\mathcal{S}(G)$ the complemented of the space $\mathcal{S}(V)\ \otimes C^\infty(K)$ tensor product of $\mathcal{S}(V)$ and $C^\infty(K)$. The topology of the space $\mathcal{S}(G)$ which is defined by the family of semi-normas

$$\partial^l_{\alpha,\beta}(f) = \sup_{|\alpha\leq p|,} \ \sup_{(v,y)\in V\times K} (1+|v|^2)^\beta \left\| Q^\alpha_v D^l f(v,y)\right\|_2 \tag{20}$$

turns $\mathcal{S}(G)$ a Frechet space wich can be called the Schwartz space of G, where$|\ |$ signifies the norm associated to $(\,|\,)$, see [5], lemma 2 and proposition 1. and 11, chap 45]. Let $L = V \times K \times K$ be the group with law:

$$(v,x,y)(w,s,t) = (v+\rho(y)w, xs, yt) \tag{21}$$

Let $\mathcal{D}(V\times K\times K\ )$ and $C^\infty(V\times K\times K)$ be C^∞ with compact support and the space of C^∞- functions of the group L.In the same manner we define the Schwartz space $\mathcal{S}(V\times K\times K)$

Definition 3.1.. *For every function f belongs to $L^1(V\times K\times K)$, one can define the Fourier transform of f by the following manner*

$$\mathcal{F}f(\xi,\gamma) = \int_V \int_K f(v,x)e^{-\ i\langle\ \xi,\ v\rangle}\ \gamma(x^{-1})dvdx \tag{22}$$

for all $\xi \in V \simeq V^*$and for all $\gamma \in \widehat{K}$. In the following we will use the Lie group L to prove by another way the plancherel formula.

Definition 3.2. *For any $f \in \mathcal{S}(G)$, we can define an function $\widetilde{f} \in \mathcal{S}(V\times K\times K)$ as follows*

$$\widetilde{f}(v,x,y) = f(x.v.xy) \tag{23}$$

note here that the function $\widetilde{f}$ is invariant in the following sense

$$\widetilde{f}(tv, xt^{-1}, ty) = \widetilde{f}(v, x, y) \tag{24}$$

We will denote by $\mathcal{S}_K(V \times K \times K)$, $\mathcal{S}_K(V \times K \times K)$, $\mathcal{S}_K(V \times K \times K)$

Definition 3.3.. *for any two function $f \in \mathcal{S}(G)$ and $F \in \mathcal{S}(V \times K \times K)$, we can define a convolution product of f and F on G*

$$\begin{aligned} f * F(v, x, y) &= \int_G F((w, z)^{-1}(v, x, y) f(w, z) dwdz \\ &= \int_G F(z^{-1}(v - w), x, z^{-1}y) f(w, z) dwdz \end{aligned} \tag{25}$$

This leads to obtain

Lemma 3.1. *If F is invarant in sense* (24)*, then we get*

$$f * F(v, x, y) = f *_c F(v, x, y) \tag{26}$$

for every $(v, x, y) \in L$, where $*$ signifies the convolution product on G with respect the variables (v, y) and $*_c$signifies the convolution product on B with respect the variables (v, x)

Proof: Let $f \in \mathcal{S}(G)$ and $F \in \mathcal{S}_K(V \times K \times K)$, then we have

$$\begin{aligned} f * F(v, x, y) &= \int_G F((w, z)^{-1}(v, x, y) f(w, z) dwdz \\ &= \int_G F(z^{-1}(v - w), x, z^{-1}y) f(w, z) dwdz \\ &= \int_B F((v - w), xz^{-1}, y) f(w, z) dwdz \end{aligned} \tag{27}$$

So the lemma is proved.

Definition 3.4. *If $f \in \mathcal{S}(G)$, one defines the Fourier transform of its invariant $\widetilde{f}$ as follows*

$$\mathcal{F}\widetilde{f}(\xi, \gamma, 1) = \int_V \int_K [\sum_{\delta \in \widehat{K}} d_\delta \int_K tr(\Im \widetilde{f}(v, x, y)\delta(y^{-1})dy)]\gamma(x^{-1})dx e^{-i\langle \xi, v\rangle} dv \tag{28}$$

where $\Im$ is the partial Fourier transform on the compact Lie group.

Theorem 3.1. *For any two functions g and f belong to G , the we have*

$$\mathcal{F}(g * \widetilde{f})(\xi,\gamma,1) = \mathcal{F}(g)(\xi,\gamma)\mathcal{F}(\widetilde{f})(\xi,\gamma,1) \tag{29}$$

Proof: By lemma 3.1. we have if f and g two functions from $\mathcal{S}(G)$

$$\begin{aligned} & \mathcal{F}(g * \widetilde{f})(\xi,\gamma,1) \\ = & \int\limits_V \int_K \sum_{\delta\in\widehat{K}} d_\delta tr[(g * \widetilde{f})(v,x,\delta)]\gamma(x^{-1})dxe^{-\,i\langle\,\xi,\,v\rangle}dv \quad (30) \\ = & \int\limits_V \int_K \sum_{\delta\in\widehat{K}} d_\delta tr[\int_K ((g * \widetilde{f})(v,x,y))\delta(y^{-1})dy]\gamma(x^{-1})dxe^{-\,i\langle\,\xi,\,v\rangle}dv \quad (31) \\ = & \int\limits_V \int_K \sum_{\delta\in\widehat{K}} d_\delta tr[\int_K ((g *_c \widetilde{f})(v,x,y))\delta(y^{-1})dy]\gamma(x^{-1})dxe^{-\,i\langle\,\xi,\,v\rangle}dv \quad (32) \end{aligned}$$

Chinging variables $v - u = w, xt^{-1} = z$, this implies

$$\begin{aligned} & \mathcal{F}(g * \widetilde{f})(\xi,\gamma,1) \\ = & \int\limits_V \int_K \int\limits_V \int_K \widetilde{f}(v-u, xt^{-1}, 1)]g(u,t)dudt\gamma(x^{-1})dxe^{-\,i\langle\,\xi,\,v\rangle}dv \\ = & \int\limits_V \int_K \int\limits_V \int_K \widetilde{f}(w,z,1)]g(u,t)\gamma(t^{-1}z^{-1})dxe^{-\,i\langle\,\xi,\,w+u\rangle}dudtdwdzvdx \\ = & \int\limits_V \int_K \int\limits_V \int_K \widetilde{f}(w,z,1)]g(u,t)\gamma(z^{-1})\gamma(t^{-1})dzdte^{-\,i\langle\,\xi,\,w\rangle}e^{-\,i\langle\,\xi,\,u\rangle}dudw \\ = & \mathcal{F}\widetilde{f}(\xi,\gamma,1)\mathcal{F}g(\xi,\gamma) \quad (33) \end{aligned}$$

Theorem 3.2.(*Plancheral's formula*) *For any $f \in L^1(G) \cap L^2(G)$,we get*

$$\begin{aligned} f * \overset{\vee}{f}(0,1) &= \int\limits_G |f(v,x)|^2 \, dvdx \\ &= \sum_{\gamma\in\widehat{K}} d_\gamma \int\limits_V \|\mathcal{F}f(\xi,\gamma)\|_2^2 \, d\xi \quad (34) \end{aligned}$$

Proof: First, let $\overset{\sim}{\overset{\vee}{f}}$ be the function defined by

$$\overset{\sim}{\overset{\vee}{f}}(v,x,y) = \overline{f((xv,xy)^{-1})} \tag{35}$$

then we have

$$f * \overset{\sim}{\overset{\vee}{f}}(0,1,1) = \int_V \sum_{\gamma\in\widehat{K}} \sum_{\delta\in\widehat{K}} d_\gamma d_\delta tr(\mathcal{F}(f * \overset{\sim}{\overset{\vee}{f}})(\xi,\gamma,\delta))d\xi$$

$$\begin{aligned}
&= \int_V \sum_{\gamma\in\widehat{K}} d_\gamma tr\{[\sum_{\delta\in\widehat{K}} d_\delta tr[\Im\mathcal{F}\overset{\sim}{\overset{\vee}{f}}(\xi,\gamma,\delta)]\mathcal{F}f(\xi,\gamma))\}d\xi \\
&= \sum_{\gamma\in\widehat{K}} d_\gamma \int_V tr[\mathcal{F}\overset{\sim}{\overset{\vee}{f}}(\xi,\gamma,1)\mathcal{F}f(\xi,\gamma)]d\xi = \sum_{\gamma\in\widehat{K}} d_\gamma \int_V tr[\overline{\mathcal{F}f(\xi,\gamma^t)}\mathcal{F}f(\xi,\gamma)]d\xi \\
&= \sum_{\gamma\in\widehat{K}} d_\gamma \int_V tr[(\mathcal{F}f(\xi,\gamma))^*\mathcal{F}f(\xi,\gamma)]d\xi \\
&= \sum_{\gamma\in\widehat{K}} d_\gamma \int_V \|\mathcal{F}f(\xi,\gamma)\|_2^2 \, d\xi = \int_G |f(v,x)|^2 \, dvdx
\end{aligned} \tag{36}$$

4 New Groups.

In this section I will introduce new group whose names are not known for Mathematicians and Physicians. But they will be very useful.

4.1. The first new group is: $\mathbb{R}_-^\star = \{x \in \mathbb{R};\ x \langle 0\ \}$, with law

$$x \cdot y = -x\ .y \tag{37}$$

for all $x \in \mathbb{R}_-^\star$ and $y \in \mathbb{R}_-^\star$, where $\cdot$ signifies the product in $\mathbb{R}_-^\star$ and . signifies the ordinary product of two real numbers.

Theorem 4.1. $(\mathbb{R}_-^\star,\ \cdot)$ *with the law* $\cdot$ *becomes a commutative group isomorphic with the multiplicative group* $(\mathbb{R}_+^\star,\ .)$ *and with the real vector group* $(\mathbb{R}, +)$.

Proof: the identity element is -1 because

$$\begin{aligned}
(-1)\cdot x &= -(-1)\cdot x = x, \ \textit{and} \\
x\cdot(-1) &= -x\ .(-1)
\end{aligned} \tag{38}$$

If $(x\ ,y) \in \mathbb{R}_{-}^{\star} \times \mathbb{R}_{-}^{\star}$, such that $x \cdot y = -1$, then we get

$$x \cdot y = -x\ .y = -1 \tag{39}$$

From this equation we obtain $y = x^{-1} = \frac{1}{x} \in \mathbb{R}_{-}^{\star}$, which is the inverse of x. Now let x, y, and z be three elements belong to $\mathbb{R}_{-}^{\star}$, then we have

$$(x \cdot y) \cdot z = -(x \cdot y).z = -(-x\ .y).z = x\ .y.z \tag{40}$$

and

$$x \cdot (y \cdot z) = x \cdot (-y.z) = -x.(-y.z) = x.y.z \tag{41}$$

So the law is associative and clearly commutative.

Let $\psi : \mathbb{R}_{-}^{\star} \to \mathbb{R}_{+}^{\star}$ the mapping defined by

$$\psi(x) = -x \tag{42}$$

then we get

$$\begin{aligned} \psi(x \cdot y) &= -(x\ \cdot y) \\ &= x.y = (-x).(-y) = \psi(x).\psi(y) \end{aligned} \tag{43}$$

That means ψ is homomorphism from $\mathbb{R}_{-}^{\star}$ to $\mathbb{R}_{+}^{\star}$ and evidently is one-to-one and surjective, and so ψ is a group isomorphism from $\mathbb{R}_{-}^{\star}$ onto $\mathbb{R}_{+}^{\star}$. Then $\mathbb{R}^{\star} = \mathbb{R}_{-}^{\star} \cup \mathbb{R}_{+}^{\star} = \mathbb{R}_{+}^{\star}\ \cup \mathbb{R}_{+}^{\star}$

Definition 4.1. *Let f belongs to $\mathcal{D}(\mathbb{R}_{-}^{\star})$, on can define the Fourier transform of by*

$$\mathcal{F}f(\lambda) = \int\limits_{\mathbb{R}_{-}^{\star}} f(y)(-y)^{-i\lambda}\ \frac{dy}{y} \tag{44}$$

Corollary 4.1. *For any* $f \in L^1(R_{-}^{\star}) \cap\ L^2(R_{-}^{\star})$, **we get**

$$\int\limits_{\mathbb{R}_{-}^{\star}} |f(x)|^2 \frac{dx}{x} = \int\limits_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2\, d\lambda \tag{45}$$

4.2. The second group is: $GL_{-}(n,\mathbb{R})$: *Let $GL(n,\mathbb{R})$ be the general linear group consisting of all matrices of the form*

$$GL(n,\mathbb{R}) = \{X = \left(\ a_{ij}\ \right), a_{ij} \in \mathbb{R},\ 1\langle i,\ j\ \langle n,\ and\ \det A \neq 0\} \tag{46}$$

and let be the subset $GL_{-}(n,\mathbb{R})$ of $GL(n,\mathbb{R})$, which is defined as

$$GL_{-}(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}),\ \det A\ \langle 0\ \} \tag{47}$$

Definition 4.2. *We supply $GL_{-}(n,\mathbb{R})$ by the law noted $\cdot$following structure*

$$A \bullet B = I_{-}A.B \tag{48}$$

for any $A \in GL_{-}(n,\mathbb{R})$ and $B \in GL_{-}(n,\mathbb{R})$, where $\cdot$ signifies the multiplication in $GL_{-}(n,\mathbb{R})$ and . signifies the usual multiplication of two matrix and I_{-} is the the following matrix defined as

$$(a_{ij}) \tag{49}$$

where $a_{11} = -1$ and $a_{ii} = 1$ for any $1\langle i \leq n$, and $a_{ij} = 0,\ i \neq j$.

Theorem: (i) $(GL_{-}(n,\mathbb{R}),\ \bullet)$ *is group, and is isomorphic onto the subgroup* $GL_{+}(n,\mathbb{R})$

Proof: It sufficies to consider the mapping

$$\varphi(A) = I_{-}A,\ \forall A \in GL_{-}(n,\mathbb{R}) \tag{50}$$

4.3. The third new group is: $O_{-}(n,\mathbb{R})$: Let $O(n,\mathbb{R})$ be the orthogonal Lie group consisting of all matrices of the form

$$O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}),\ AA^{*} = I\ \} \tag{51}$$

where I is the identity matrix. Let $O_{-}(n,\mathbb{R})$ be the subset of $O(n,\mathbb{R})$ that is defined by

$$O_{-}(n,\mathbb{R}) = \{A \in O(n,\mathbb{R}),\ \det A\ \langle 0\ \} \tag{52}$$

It is easy to show that $(O_{-}(n,\mathbb{R}),\ \bullet)$ becomes group isomorphic onto $SO(n,\mathbb{R}$.

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