Octonion Representation of the Superstring Theory

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Abstract

In this paper, we have made an attempt to discuss the role of octonions in superstring theory (*i.e.* a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong) where, we have described the octonion representation of the super-string (SS) formulation as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions. We have discussed the octonionic differential operator, octonionic valued potential wave equation, octonionic field equation and other various quantum equations of superstring formulation in simpler, compact and consistent manner. Consequently, the generalized Dirac-Maxwell's equations are studied with the preview of superstring theory by means of octonions.

Key words: octonion, superstring theory, generalized Dirac-Maxwell's equations, wave equation.

1 Introduction

Physics is a natural science that involves [1] the study of matter and its motion through space and time, along with related concepts such as energy and force. The search for unity and simplicity has been the theme of physics ever since Newton first showed that celestial and terrestrial mechanics could be unified . The 20^{th} century has been a time for tremendous format and change in our understanding new phenomena or adding new features to the existing theories [2]. Particle physics [1, 2] is the branch of physics which deals with the study of matter, energy, space and time. Its objectives are to identify the most simple objects out of which all matter is composed and to understand the forces which cause them to interact and combine to make more complex things. Now a days particle physics is popularly known as high energy physics [1, 2, 3], which is the theory of basic structure of matter and its forces. The physics of elementary particles is currently described in terms of very successful theory called standard model [1, 4]. It describes all known elementary particles and their interactions except gravitational interactions. The standard model accommodates the quarks and the leptons, which are constituents of matter, the vector particles that mediate the strong and electroweak forces and Higgs

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bosons, which is expected to account for the masses of particles. The standard model (SM) also describes [5] the unified picture of strong and electro-weak interactions within the framework of a $SU(3) \times SU(2) \times U(1)$ non-Abelian gauge theory.

Physics beyond the Standard Model [6, 7] refers to the theoretical developments needed to explain the deficiencies of the Standard Model, such as the origin of mass, the strong CP problem, neutrino oscillations, matter–antimatter asymmetry, and the nature of dark matter and dark energy [6, 8]. Another problem lies within the mathematical framework of the Standard Model itself. The Standard Model is inconsistent with that of general relativity to the point that one or both theories break down in their descriptions under certain conditions (for example within known space-time singularities like the Big Bang and black hole event horizons).

A Grand Unified Theory (GUT) [6, 7], is a model in particle physics in which at high energy, the three gauge interactions of the Standard Model which define the electromagnetic, weak, and strong interactions, are merged into one single interaction characterized by one larger gauge symmetry and thus one unified coupling constant. In contrast, the experimentally verified Standard Model of particle physics is based on three independent interactions, symmetries and coupling constants. The standard model has three gauge symmetries [9]; the colour $SU(3)$, the weak isospin $SU(2)$, and the hypercharge $U(1)$ symmetry, corresponding to the three fundamental forces. Due to renormalization the coupling constants of each of these symmetries vary with the energy at which they are measured. Around 10^{16} GeV these couplings become approximately equal. This has led to speculation that above this energy the three gauge symmetries of the standard model are unified in one single gauge symmetry with a simple group gauge group, and just one coupling constant. Below this energy the symmetry is spontaneously broken to the standard model symmetries [10]. Unifying gravity with the other three interactions would provide a theory of everything (TOE), rather than a GUT. Nevertheless, GUTs are often seen as an intermediate step towards a TOE. The new particles predicted by models of grand unification cannot be observed directly at particle colliders because their masses are expected to be of the order of the so-called GUT scale, which is predicted to be just a few orders of magnitude below the Planck scale and thus far beyond the reach of currently foreseen collision experiments. Instead, effects of grand unification might be detected through indirect observations such as proton decay, electric dipole moments of elementary particles, or the properties of neutrinos [11]. Some grand unified theories predict the existence of magnetic monopoles. Popular choices for the unifying group are the special unitary group in five dimensions SU(5) and the special orthogonal group in ten dimensions SO(10) [12]. String theory [13] is one such reinvention, and many theoretical physicists think that such theories are the next theoretical step toward a true Theory of Everything. Theories of quantum gravity such as loop quantum gravity and others are thought by some to be promising candidates to the mathematical unification of quantum field theory and general relativity, requiring less drastic changes to existing theories [14]. However recent work places stringent limits on the putative effects of quantum gravity on the speed of light, and disfavors some current models of quantum gravity [15].

On the other hand, there has been a revival in the formulation of natural laws so that there exists [16] four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathscr{O}$). All four algebra's are alternative with totally anti symmetric associators. Quaternions [17, 18] were very first example of hyper complex numbers have been widely used [19]-[25] to the various applications of mathematics and physics. Since octonions [26] share with complex numbers and quaternions, many attractive mathematical properties, one might except that they would be equally as useful as others. Octonion analysis has been widely discussed by Baez [27]. It has also played an important role in the context of various physical problems [28, 29, 30, 31] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [32, 33, 34, 35] towards the developments of wave equation and octonion form of Maxwell's equations. We [36, 37, 38] have also studied generalized octonion electrodynamics, generalized split octonion electrodynamics and octonion quantum chromodynamics and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation. In this paper, we have made an attempt to discuss the role of octonions in superstring theory (*i.e.* a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong) where, we have described the octonion representation of the super-string (SS) formulation as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions. We have discussed the octonionic differential operator, octonionic valued potential wave equation, octonionic field equation and other various quantum equations of superstring formulation in simpler, compact and consistent manner.

2 Mathematical Preliminaries

An octonions (Cayley numbers) is the family of hypercomplex numbers and have eight dimensions. Thus, octonion $x \in \mathscr{O}$ is expressed [39, 40] as

$$\begin{aligned} x &= e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 \\ &= e_0x_0 + \sum_{A=1}^{7} e_A x_A \end{aligned} \tag{1}$$

where $e_A (A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. Set of octets $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis elements and satisfy the following multiplication rules

$$\begin{aligned} &e_0 = 1;\, e_0e_A = e_Ae_0 = e_A; \\ &e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C.\ (A, B, C = 1, 2, ..., 7). \end{aligned} \tag{2}$$

The structure constants f_{ABC} is completely antisymmetric and takes the value 1 for following combinations,

$$f_{ABC} = +1; \forall (ABC) = (123), (471), (257), (165), (624), (543), (736). \tag{3}$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra $\mathscr{O}$ is described over the algebra of real numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by equations (2) and (3) are then generalized in the following table:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Table 1: Octonion Multiplication table

As such we may write the following relations among octonion basis elements

$$\begin{aligned} [e_A, e_B] &= 2 f_{ABC} e_C; \\ \{e_A, e_B\} &= -2 \delta_{AB} e_0; \\ e_A(e_B e_C) &\neq (e_A e_B) e_C. \end{aligned} \tag{4}$$

Octonion conjugate is defined as

$$\begin{aligned} \bar{x} &= e_0 x_0 - e_1 x_1 - e_2 x_2 - e_3 x_3 - e_4 x_4 - e_5 x_5 - e_6 x_6 - e_7 x_7 \\ &= e_0 x_0 - \sum_{A=1}^{7} e_A x_A \end{aligned} \tag{5}$$

where we have used the conjugates of basis elements as $\overline{e_0} = e_0$ and $\overline{e_A} = -e_A$. Hence an octonion can be decomposed in terms of its scalar $(Sc(x))$ and vector $(Vec(x))$ parts as

$$\begin{aligned} Sc(x) &= \frac{1}{2}(x + \bar{x}); \\ Vec(x) &= \frac{1}{2}(x - \bar{x}) = \sum_{A=1}^{7} e_A x_A. \end{aligned} \tag{6}$$

Conjugates of product of two octonions and its own are described as

$$\overline{(xy)} = \bar{y}\,\bar{x}; \qquad \overline{(\bar{x})} = x. \tag{7}$$

while the scalar product of two octonions is defined as

$$\langle x, y \rangle = \frac{1}{2}(x\bar{y} + y\bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x) = \sum_{\alpha=0}^{7} x_\alpha y_\alpha. \tag{8}$$

The norm $N(x)$ and inverse x^{-1} (for a nonzero x) of an octonion are respectively defined as

$$N(x) = x\bar{x} = \bar{x}x = \sum_{\alpha=0}^{7} x_\alpha^2 . e_0; \;\; x^{-1} = \frac{\bar{x}}{N(x)} \Longrightarrow x x^{-1} = x^{-1} x = 1. \tag{9}$$

The norm $N(x)$ of an octonion x is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x). \tag{10}$$

Equation (4) shows that octonions are not associative in nature and thus do not form the group in their usual form. Non - associativity of octonion algebra $\mathscr{O}$ is provided by the associator $(x,y,z) = (xy)z - x(yz)\ \forall x,y,z \in \mathscr{O}$ defined for any three octonions. If the associator is totally antisymmetric for exchanges of any three variables, i.e. $(x,y,z) = -(z,y,x) = -(y,x,z) = -(x,z,y)$, then the algebra is called alternative.

3 Homogeneous and Inhomogeneous Octonion Wave Equation

In order to consider the properties of octonions [41, 42] and its eight dimensional connection, we may now write the octonion differential operator $\mathscr{D}$ [43, 44] as

$$\mathscr{D}\{\partial_0, \partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7\} = \sum_{\mu=0}^{7} e_\mu \partial_\mu, \qquad (\mu = 0,1,2,3,......,7) \tag{11}$$

where ∂_μ are described as the components of a differential operator in an eight dimensional representation. Here we may consider the eight dimensional space as the combination of two (external and internal) four dimensional spaces. As such, a function of an octonion variable may be described as

$$\mathscr{F}(X)\{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7\} = \sum_{\mu=0}^{7} e_\mu f_\mu(X), \tag{12}$$

Here f_μ are scalar functions. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions [45]. As such, a function $\mathscr{F}(X)$ of an octonion variable $X\{X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7\} = \sum_{\mu=0}^{7} e_\mu X_\mu$ is left regular at X if and only if $\mathscr{F}(X)$ satisfies the condition

$$\mathscr{D}\mathscr{F}(X) = 0. \tag{13}$$

Similarly, a function $G(X)\{g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ is a right regular if and only if

$$G(X)\mathscr{D} = 0, \tag{14}$$

where $G(X) = g_0 + g_1 e_1 + g_2 e_2 + ... + g_7 e_7$. Then we get

$$\mathscr{D}\mathscr{F} = I\{I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7\} = \sum_{\mu=0}^{7} e_\mu I_\mu, \tag{15}$$

where

$$\begin{aligned}
I_0 &= \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 - \partial_3 f_3 - \partial_4 f_4 - \partial_5 f_5 - \partial_6 f_6 - \partial_7 f_7;\\
I_1 &= \partial_0 f_1 + \partial_1 f_0 + \partial_2 f_3 - \partial_3 f_2 + \partial_6 f_5 - \partial_5 f_6 - \partial_7 f_4 + \partial_4 f_7;\\
I_2 &= \partial_0 f_2 + \partial_2 f_0 + \partial_3 f_1 - \partial_1 f_3 + \partial_4 f_6 - \partial_6 f_4 - \partial_7 f_5 + \partial_5 f_7;\\
I_3 &= \partial_0 f_3 + \partial_3 f_0 + \partial_1 f_2 - \partial_2 f_1 + \partial_6 f_7 - \partial_7 f_6 + \partial_5 f_4 - \partial_4 f_5;\\
I_4 &= \partial_0 f_4 + \partial_4 f_0 + \partial_3 f_5 - \partial_5 f_3 - \partial_2 f_6 + \partial_6 f_2 - \partial_1 f_7 + \partial_7 f_1;\\
I_5 &= \partial_0 f_5 + \partial_5 f_0 + \partial_1 f_6 - \partial_6 f_1 + \partial_7 f_2 - \partial_2 f_7 - \partial_3 f_4 + \partial_4 f_3;\\
I_6 &= \partial_0 f_6 + \partial_6 f_0 - \partial_1 f_5 + \partial_5 f_1 + \partial_2 f_4 - \partial_4 f_2 - \partial_3 f_7 + \partial_7 f_3;\\
I_7 &= \partial_0 f_7 + \partial_7 f_0 + \partial_1 f_4 - \partial_4 f_1 + \partial_2 f_5 - \partial_5 f_2 - \partial_6 f_3 + \partial_3 f_6.
\end{aligned} \tag{16}$$

The regularity condition (13) may now be considered as a homogeneous octonion wave equation for octonion variables without sources. On the other hand, equation (15) is considered as the inhomogeneous wave equation as

$$\mathscr{D}\mathscr{F} = I. \tag{17}$$

where I is also an octonion. Similarly, we may also write the homogeneous as well as inhomogeneous octonion wave equations on using the right regularity condition (14). We may now interpret these octonion wave equations as the classical wave (field) equations of physical variables. Thus, one dimensional octonion representation is identical to eight dimensional spaces over the field of real numbers.

4 Octonion form of Generalized Dirac-Maxwell's (GDM) Equations

In order to consider the generalized electromagnetic fields of dyon, we may write the various quantum equations of dyons in octonion formulation. Thus the octonion valued potential, in eight dimensional formalism as the combinations of two four dimensional spaces, is defined as

$$\mathbb{V}\{V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7\} = \sum_{\mu=0}^{7} e_\mu V_\mu, \qquad (\mu = 0, 1, 2, 3, ..., 7). \tag{18}$$

We may now identify the components of generalized potential of dyons as

$$\begin{aligned}
&V_0 \longmapsto \varphi, \;\; V_1 \longmapsto A_x, \; V_2 \longmapsto A_y, \;\; V_3 \longmapsto A_z,\\
&V_4 \longmapsto iB_x, \;\; V_5 \longmapsto iB_y, \; V_6 \longmapsto iB_z, \;\; V_7 \longmapsto i\phi \;\; (i = \sqrt{-1})
\end{aligned} \tag{19}$$

where $(\phi, A_x, A_y, A_z) = (\phi, \overrightarrow{A}) \equiv \{A^\mu\}$ and $(\varphi, B_x, B_y, B_z) = (\varphi, \overrightarrow{B}) \equiv \{B^\mu\}$ are respectively described as the components of electric $\{A_\mu\}$ and magnetic $\{B_\mu\}$ four potentials of dyons. In order to obtain the generalized field equations of dyons in four dimensional space time, we may identify differential operator () to be four dimensional, so that the differential operator (11) be written as

$$\mathscr{D} \longmapsto \boxdot\{e_1, e_2, e_3, e_7\} \;\; = \;\; e_j \partial_j + e_7 \partial_7, \tag{20}$$

where $\partial_7 = -i\frac{\partial}{\partial t}$ $(i = \sqrt{-1})$, $\partial_j = \frac{\partial}{\partial x_j}$ $(j = 1,2,3)$ and other components may be taken vanishing as we are concerned with classical electrodynamics of dyons in four dimensional space-time world [36, 37]. Octonion conjugate of equation (20) may then be written as

$$\overline{\mathscr{D}} \longmapsto \overline{\boxdot}\{e_1, e_2, e_3, e_7\} = -e_j\partial_j - e_7\partial_7, \tag{21}$$

Now operating $\overline{\boxdot}$ given by equation (21) to octonion potential $\mathbb{V}$ of equation (18) for the octonionic potential wave equations, we get

$$\overline{\boxdot}\mathbb{V} = \mathbb{F}\{F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7\} = \sum_{\mu=0}^{7} e_\mu F_\mu, \qquad (\mu = 0, 1, 2, 3, ..., 7). \tag{22}$$

where $\mathbb{F}$ is an octonion reproduces the generalized electromagnetic fields of dyons, may be express as

$$\begin{aligned} F_1 = (-\partial_1\varphi + \partial_2 A_z - \partial_3 A_y - i\partial_7 B_x) \mapsto H_x; \quad F_2 = (-\partial_2\varphi + \partial_3 A_x - \partial_1 A_z - i\partial_7 B_y) \mapsto H_y; \\ F_3 = (-\partial_3\varphi + \partial_1 A_y - \partial_2 A_x - i\partial_7 B_z) \mapsto H_z; \quad F_4 = i(-\partial_1\phi - \partial_2 B_z + \partial_3 B_y - i\partial_7 A_x) \mapsto iE_x; \\ F_5 = i(-\partial_2\phi - \partial_3 B_x + \partial_1 B_z - i\partial_7 A_y) \mapsto iE_y; \quad F_6 = i(-\partial_3\phi - \partial_1 B_y + \partial_2 B_x - i\partial_7 A_z) \mapsto iE_z; \\ F_0 = -(\overrightarrow{\nabla}\cdot\overrightarrow{A} + i\partial_7\phi) \mapsto 0; \quad F_7 = i(\overrightarrow{\nabla}\cdot\overrightarrow{B} + i\partial_7\varphi) \mapsto 0. \end{aligned} \tag{23}$$

Here $\overrightarrow{E}\,(E_x, E_y, E_z)$ and $\overrightarrow{H}\,(H_x, H_y, H_z)$ are the components of electric and magnetic four potentials. The generalized electric $(\overrightarrow{E})$ and magnetic $(\overrightarrow{H})$ fields of dyons in terms of components of electric and magnetic four potentials

$$\begin{aligned} \overrightarrow{E} &= -\frac{\partial\overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\phi - \overrightarrow{\nabla}\times\overrightarrow{B}; \\ \overrightarrow{H} &= -\frac{\partial\overrightarrow{B}}{\partial t} - \overrightarrow{\nabla}\varphi + \overrightarrow{\nabla}\times\overrightarrow{A}. \end{aligned} \tag{24}$$

Now applying the differential operator (20) to equation (22), we get

$$\boxdot\mathbb{F} = \mathbb{J}\{J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7\} = \sum_{\mu=0}^{7} e_\mu J_\mu, \qquad (\mu = 0, 1, 2, 3, ..., 7).; \tag{25}$$

We may now identify the components of generalized current of dyons as

$$\begin{aligned} J_0 \longmapsto -\rho, \;\; J_1 \longmapsto \mathrm{j_x}, \; J_2 \longmapsto \mathrm{j_y}, \;\; J_3 \longmapsto \mathrm{j_z}, \\ J_4 \longmapsto -ik_x, \;\; J_5 \longmapsto -ik_y, \; J_6 \longmapsto -ik_z, \;\; J_7 \longmapsto i\rho \;\; (i = \sqrt{-1}) \end{aligned} \tag{26}$$

Here $(\rho, \overrightarrow{j}) = \{\mathrm{j}_\mu\}$ and $(\rho, \overrightarrow{j}) = \{k_\mu\}$ are respectively the four currents associated with electric charge and magnetic monopole of dyons. Generalized Dirac-Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. Thus, equations (25) and (26) thus lead to following differential equations,

$$\begin{aligned}(\overrightarrow{\nabla}\cdot\overrightarrow{E}) &= \rho;\\ (\overrightarrow{\nabla}\times\overrightarrow{E}) &= -\frac{\partial\overrightarrow{H}}{\partial t}-\vec{k};\\ (\overrightarrow{\nabla}\times\overrightarrow{H}) &= \frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j};\\ (\overrightarrow{\nabla}\cdot\overrightarrow{H}) &= \rho.\end{aligned} \tag{27}$$

which are the generalized Dirac-Maxwell's (GDM) equations of generalized fields of dyons. Like quaternion formulation of generalized electromagnetic fields of dyons. Octonion formulation is compact and simpler. Since e_7 is coupling two quaternions into one octonion and also reverses its sign in its combination with quaternion units.

5 Octonion Superstring Formulation

In the early days, one of the grate attractions of superstring theory was that it appeared to be uniquely determined - we live, it seemed, in the only mathematically possible world. The superstring theory still holds out the best hope for ultimate unification of all four interactions (i.e. gravitational, electromagnetic, weak and strong interactions), and it is probably the most promissing candidate for quantum gravity. Thus, the octonionic representation of the super-string (SS) formulation may be consider as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions [46, 47], *i.e.* unification of the four fundamental forces. So, we may write the octonionic superstring space as

$$\begin{aligned}\mathscr{O}_{SS} =&\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\\ =&(\mathscr{O}_g,\mathscr{O}_{em},\mathscr{O}_w,\mathscr{O}_s)\Longrightarrow((e_0,e_1),(e_2,e_3),(e_4,e_5),(e_6,e_7)),\end{aligned} \tag{28}$$

where $\mathscr{O}_g$, $\mathscr{O}_{em}$, $\mathscr{O}_w$, $\mathscr{O}_s$ are respectively known as gravitational, electromagnetic, weak and strong spaces in superstring theory related with the octonionic basis $(e_0,e_1),(e_2,e_3),(e_4,e_5),(e_6,e_7)$. Thus, the octonionic physical quantity $\mathbb{X}\in\mathscr{O}_{SS}$ is expressed as

$$\begin{aligned}\mathbb{X} =&X_g+X_{em}+X_w+X_s\\ =&(X_{g_0}e_0+X_{g_1}e_1)+(X_{em_0}e_2+X_{em_1}e_3)+(X_{w_0}e_4+X_{w_1}e_5)+(X_{s_0}e_6+X_{s_1}e_7).\end{aligned} \tag{29}$$

The octonionic differential operator in case of superstring formulation (i.e. unification of four differential operator) may be written as

$$\begin{aligned}\boxplus_{SS} =&\boxdot_g+\boxdot_{em}+\boxdot_w+\boxdot_s\\ =&(\partial_{g_0}e_0+\partial_{g_1}e_1)+(\partial_{em_0}e_2+\partial_{em_1}e_3)+(\partial_{w_0}e_4+\partial_{w_1}e_5)+(\partial_{s_0}e_6+\partial_{s_1}e_7).\end{aligned} \tag{30}$$

Octonionic conjugate of equation (30) is described as

$$\overline{\boxplus}_{SS} = (\partial_{g_0} e_0 - \partial_{g_1} e_1) - (\partial_{em_0} e_2 + \partial_{em_1} e_3) - (\partial_{w_0} e_4 + \partial_{w_1} e_5) - (\partial_{s_0} e_6 + \partial_{s_1} e_7). \tag{31}$$

Thus, the octonionic superstring valued potential with the combination of four potentials may be expressed as

$$\begin{aligned}\mathbb{V}_{SS} = (V_g\,, V_{em}\,, V_w\,, V_s) &= ((V_0, V_1)\,, (V_2, V_3)\,, (V_4, V_5)\,, (V_6, V_7)) \\ &= \left((V_{g_0}, V_{g_1})\,, (V_{em_0}, V_{em_1})\,, (V_{w_0}, V_{w_1})\,, (V_{s_0}, V_{s_1})\right),\end{aligned} \tag{32}$$

which is further simplified to

$$\mathbb{V}_{SS} = (V_{g_0} e_0 + V_{g_1} e_1) + (V_{em_0} e_2 + V_{em_1} e_3) + (V_{w_0} e_4 + V_{w_1} e_5) + (V_{s_0} e_6 + V_{s_1} e_7). \tag{33}$$

In order to obtain the octonionic potential wave equations of the superstring space, let us operate $\overline{\boxplus}_{SS}$ given by equation (31) to octonion superstring valued potential $\mathbb{V}_{SS}$ of equation (33) and we get

$$\begin{aligned}\overline{\boxplus}_{SS} \mathbb{V}_{SS} = & e_0\{\partial_{g_0} V_{g_0} + \partial_{g_1} V_{g_1} + \partial_{em_0} V_{em_0} + \partial_{em_1} V_{em_1} + \partial_{w_0} V_{w_0} + \partial_{w_1} V_{w_1} + \partial_{s_0} V_{s_0} + \partial_{s_1} V_{s_1}\} \\ & + e_1\{\partial_{g_0} V_{g_1} - \partial_{g_1} V_{g_0} - \partial_{em_0} V_{em_1} + \partial_{em_1} V_{em_0} - \partial_{w_0} V_{s_1} + \partial_{w_1} V_{s_0} - \partial_{s_0} V_{w_1} + \partial_{s_1} V_{w_0}\} \\ & + e_2\{\partial_{g_0} V_{em_0} + \partial_{g_1} V_{em_1} - \partial_{em_0} V_{g_0} - \partial_{em_1} V_{g_1} - \partial_{w_0} V_{s_0} - \partial_{w_1} V_{s_1} + \partial_{s_0} V_{w_0} + \partial_{s_1} V_{w_1}\} \\ & + e_3\{\partial_{g_0} V_{em_1} - \partial_{g_1} V_{em_0} + \partial_{em_0} V_{g_1} - \partial_{em_1} V_{g_0} + \partial_{w_0} V_{w_1} - \partial_{w_1} V_{w_0} - \partial_{s_0} V_{s_1} + \partial_{s_1} V_{s_0}\} \\ & + e_4\{\partial_{g_0} V_{w_0} + \partial_{g_1} V_{s_1} + \partial_{em_0} V_{s_0} - \partial_{em_1} V_{w_1} - \partial_{w_0} V_{g_0} + \partial_{w_1} V_{em_1} - \partial_{s_0} V_{em_0} - \partial_{s_1} V_{g_1}\} \\ & + e_5\{\partial_{g_0} V_{w_1} - \partial_{g_1} V_{s_0} + \partial_{em_0} V_{s_1} + \partial_{em_1} V_{w_0} - \partial_{w_0} V_{em_1} - \partial_{w_1} V_{g_0} + \partial_{s_0} V_{g_1} - \partial_{s_1} V_{em_0}\} \\ & + e_6\{\partial_{g_0} V_{s_0} + \partial_{g_1} V_{w_1} - \partial_{em_0} V_{w_0} + \partial_{em_1} V_{s_1} + \partial_{w_0} V_{em_0} - \partial_{w_1} V_{g_1} - \partial_{s_0} V_{g_0} - \partial_{s_1} V_{em_1}\} \\ & + e_7\{\partial_{g_0} V_{s_1} - \partial_{g_1} V_{w_0} - \partial_{em_0} V_{w_1} - \partial_{em_1} V_{s_0} + \partial_{w_0} V_{g_1} + \partial_{w_1} V_{em_0} + \partial_{s_0} V_{em_1} - \partial_{s_1} V_{g_0}\},\end{aligned} \tag{34}$$

which provides the following octonionic analogous of superstring formulation as

$$\overline{\boxplus}_{SS} \mathbb{V}_{SS} \quad = \quad \mathbb{F}_{SS} = ((F_0, F_1), (F_2, F_3), (F_4, F_5), (F_6, F_7)); \tag{35}$$

where $\mathbb{F}_{SS}(F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7)$ is an octonion which reproduces the superstring field strengths. So, equation (35) may further be expressed as

$$\begin{aligned}\mathbb{F}_{SS} = F_g + F_{em} + F_w + F_s &= \left((F_{g_0}, F_{g_1})\,, (F_{em_0}, F_{em_1})\,, (F_{w_0}, F_{w_1})\,, (F_{s_0}, F_{s_1})\right) \\ &= (F_{g_0} e_0 + F_{g_1} e_1) + (F_{em_0} e_2 + F_{em_1} e_3) \\ &\quad + (F_{w_0} e_4 + F_{w_1} e_5) + (F_{s_0} e_6 + F_{s_1} e_7),\end{aligned} \tag{36}$$

where the first term ($F_g = F_{g_0}, F_{g_1}$) is defined as the gravitational field strength in G-space, the second term ($F_{em} = F_{em_0}, F_{em_1}$) is described as the electromagnetic field strength in EM-space, the third term ($F_w = F_{w_0}, F_{w_1}$) provides the weak interaction field strength in W-space and the forth term ($F_s = F_{s_0}, F_{s_1}$) is responsible for the strong field strength in S-space. Thus, the component of $\mathbb{F}_{SS}\{(F_{g_0}, F_{g_1}), (F_{em_0}, F_{em_1}), (F_{w_0}, F_{w_1}), (F_{s_0}, F_{s_1})\}$ are expressed as the following octonionic representation

$$\begin{aligned}
F_{g_0}e_0 &= \{\partial_{g_0}V_{g_0}e_0 - \partial_{g_1}V_{g_0}e_1 - \partial_{em_0}V_{g_0}e_2 - \partial_{em_1}V_{g_0}e_3 \\
&\quad - \partial_{w_0}V_{g_0}e_4 - \partial_{w_1}V_{g_0}e_5 - \partial_{s_0}V_{g_0}e_6 - \partial_{s_1}V_{g_0}e_7\}; \\
F_{g_1}e_1 &= \{\partial_{g_1}V_{g_1}e_0 + \partial_{g_0}V_{g_1}e_1 - \partial_{em_1}V_{g_1}e_2 + \partial_{em_0}V_{g_1}e_3 \\
&\quad - \partial_{s_1}V_{g_1}e_4 + \partial_{s_0}V_{g_1}e_5 - \partial_{w_1}V_{g_1}e_6 + \partial_{w_0}V_{g_1}e_7\}; \\
F_{em_0}e_2 &= \{\partial_{em_0}V_{em_0}e_0 + \partial_{em_1}V_{em_0}e_1 + \partial_{g_0}V_{em_0}e_2 - \partial_{g_1}V_{em_0}e_3 \\
&\quad - \partial_{s_0}V_{em_0}e_4 - \partial_{s_1}V_{em_0}e_5 + \partial_{w_0}V_{em_0}e_6 + \partial_{w_1}V_{em_0}e_7\}; \\
F_{em_1}e_3 &= \{\partial_{em_1}V_{em_1}e_0 - \partial_{em_0}V_{em_1}e_1 + \partial_{g_1}V_{em_1}e_2 + \partial_{g_0}V_{em_1}e_3 \\
&\quad + \partial_{w_1}V_{em_1}e_4 - \partial_{w_0}V_{em_1}e_5 - \partial_{s_1}V_{em_1}e_6 + \partial_{s_0}V_{em_1}e_7\}; \\
F_{w_0}e_4 &= \{\partial_{w_0}V_{w_0}e_0 + \partial_{s_1}V_{w_0}e_1 + \partial_{s_0}V_{w_0}e_2 - \partial_{w_1}V_{w_0}e_3 \\
&\quad + \partial_{g_0}V_{w_0}e_4 + \partial_{em_1}V_{w_0}e_5 - \partial_{em_0}V_{w_0}e_6 - \partial_{g_1}V_{w_0}e_7\}; \\
F_{w_1}e_5 &= \{\partial_{w_1}V_{w_1}e_0 - \partial_{s_0}V_{w_1}e_1 + \partial_{s_1}V_{w_1}e_2 + \partial_{w_0}V_{w_1}e_3 \\
&\quad - \partial_{em_1}V_{w_1}e_4 + \partial_{g_0}V_{w_1}e_5 + \partial_{g_1}V_{w_1}e_6 - \partial_{em_0}V_{w_1}e_7\}; \\
F_{s_0}e_6 &= \{\partial_{s_0}V_{s_0}e_0 + \partial_{w_1}V_{s_0}e_1 - \partial_{w_0}V_{s_0}e_2 + \partial_{s_1}V_{s_0}e_3 \\
&\quad + \partial_{em_0}V_{s_0}e_4 - \partial_{g_1}V_{s_0}e_5 + \partial_{g_0}V_{s_0}e_6 - \partial_{em_1}V_{s_0}e_7\}; \\
F_{s_1}e_7 &= \{\partial_{s_1}V_{s_1}e_0 - \partial_{w_0}V_{s_1}e_1 - \partial_{w_1}V_{s_1}e_2 - \partial_{s_0}V_{s_1}e_3 \\
&\quad + \partial_{g_1}V_{s_1}e_4 + \partial_{em_0}V_{s_1}e_5 + \partial_{em_1}V_{s_1}e_6 + \partial_{g_0}V_{s_1}e_7\}. \qquad (37)
\end{aligned}$$

So, in order to obtain the octonionic superstring field equations, we apply the differential operator (30) to equation (36) as

$$\begin{aligned}
&\boxplus_{SS}\mathbb{F}_{SS} = \\
&-e_0\{\partial_{g_0}F_{g_0} + \partial_{g_1}F_{g_1} + \partial_{em_0}F_{em_0} + \partial_{em_1}F_{em_1} + \partial_{w_0}F_{w_0} + \partial_{w_1}F_{w_1} + \partial_{s_0}F_{s_0} + \partial_{s_1}F_{s_1}\} \\
&+e_1\{\partial_{g_0}F_{g_1} + \partial_{g_1}F_{g_0} + \partial_{em_0}F_{em_1} - \partial_{em_1}F_{em_0} + \partial_{w_0}F_{s_1} - \partial_{w_1}F_{s_0} + \partial_{s_0}F_{w_1} - \partial_{s_1}F_{w_0}\} \\
&+e_2\{\partial_{g_0}F_{em_0} - \partial_{g_1}F_{em_1} + \partial_{em_0}F_{g_0} + \partial_{em_1}F_{g_1} + \partial_{w_0}F_{s_0} + \partial_{w_1}F_{s_1} - \partial_{s_0}F_{w_0} - \partial_{s_1}F_{w_1}\} \\
&+e_3\{\partial_{g_0}F_{em_1} + \partial_{g_1}F_{em_0} - \partial_{em_0}F_{g_1} + \partial_{em_1}F_{g_0} - \partial_{w_0}F_{w_1} + \partial_{w_1}F_{w_0} + \partial_{s_0}F_{s_1} - \partial_{s_1}F_{s_0}\} \\
&+e_4\{\partial_{g_0}F_{w_0} - \partial_{g_1}F_{s_1} - \partial_{em_0}F_{s_0} + \partial_{em_1}F_{w_1} + \partial_{w_0}F_{g_0} - \partial_{w_1}F_{em_1} + \partial_{s_0}F_{em_0} + \partial_{s_1}F_{g_1}\} \\
&+e_5\{\partial_{g_0}F_{w_1} + \partial_{g_1}F_{s_0} - \partial_{em_0}F_{s_1} - \partial_{em_1}F_{w_0} + \partial_{w_0}F_{em_1} + \partial_{w_1}F_{g_0} - \partial_{s_0}F_{g_1} + \partial_{s_1}F_{em_0}\} \\
&+e_6\{\partial_{g_0}F_{s_0} - \partial_{g_1}F_{w_1} + \partial_{em_0}F_{w_0} - \partial_{em_1}F_{s_1} - \partial_{w_0}F_{em_0} + \partial_{w_1}F_{g_1} + \partial_{s_0}F_{g_0} + \partial_{s_1}F_{em_1}\} \\
&+e_7\{\partial_{g_0}F_{s_1} + \partial_{g_1}F_{w_0} + \partial_{em_0}F_{w_1} + \partial_{em_1}F_{s_0} - \partial_{w_0}F_{g_1} - \partial_{w_1}F_{em_0} - \partial_{s_0}F_{em_1} + \partial_{s_1}F_{g_0}\}, \qquad (38)
\end{aligned}$$

which can further be reduced and be written in following compact notation in terms of an octonionic superstring representation as

$$\boxplus_{SS}\mathbb{F}_{SS} \quad = \quad \mathbb{J}_{SS} = ((J_0, J_1), (J_2, J_3), (J_4, J_5), (J_6, J_7)). \qquad (39)$$

Here $\mathbb{J}_{SS}(J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7)$ is an octonionic superstring current source, which may be expressed in the following matrix form,

$$\begin{pmatrix} J_{(g-g)} & J_{(g-g)} & J_{(g-em)} & J_{(g-em)} & J_{(g-w-s)} & J_{(g-w-s)} & J_{(g-s-w)} & J_{(g-s-w)} \\ J_{(em-em)} & J_{(em-em)} & J_{(em-g)} & J_{(em-g)} & J_{(em-s-w)} & J_{(em-s-w)} & J_{(em-w-s)} & J_{(em-w-s)} \\ J_{(w-w)} & J_{(w-s)} & J_{(w-s)} & J_{(w-w)} & J_{(w-g-em)} & J_{(w-em-g)} & J_{(w-em-g)} & J_{(w-g-em)} \\ J_{(s-s)} & J_{(s-w)} & J_{(s-s)} & J_{(s-w)} & J_{(s-em-g)} & J_{(s-g-em)} & J_{(s-g-em)} & J_{(s-em-g)} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix} \Longrightarrow \mathbb{J}_{SS}. \tag{40}$$

The generalized Dirac-Maxwell's (GDM) is visualized in superstring formulation by means of octonions and the generalized current has been discussed in equation (39). The generalized superstring current source described the various terms where $J_{(g-g)}, J_{(em-em)}, J_{(em-g)}, J_{(g-em)}$ defines as the octonionic representation of gravitational - gravitational, electromagnetic - electromagnetic, electromagnetic - gravitational, gravitational - electromagnetic current source and $J_{(w-w)}, J_{(s-s)}, J_{(s-w)}$, $J_{(w-s)}$ sectors of the octonionic current source respectively associated the weak-weak, strong-strong, strong-weak, weak strong interactions. Consequently, the remaining terms of equation (40) denotes as the parts of octonionic current sources for the combination of different three interactions. Thus, the present formulation has been provides the superstring theory in the terms of octonionic representations.

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