On Certain Subclasses of Meromorphic Univalent Funtions with Positive Coefficients

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Abstract:- **In this paper we have introduced a subclass $A(n,q,\lambda,\alpha)$ of meromorphic univalent functions with positive coefficients in the punctured unit disk $U^* = \{z \in C : 0 < |z| < 1\}$. Coefficient estimate, distortion theorem, radii of starlikeness and convexity, closure theorems and Hadamard product of functions belonging to this class are obtained. Further properties using integral operators are also obtained for the same class.**

Let Σ denote the class of meromorphic functions in the punctured unit disk $U^* = \{z \in C : 0<|z|<1\}$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

$$(a_k \geq 0 ; k \in N = \{1,2,..\}) \qquad \ldots(5.1.1)$$

For each $f(z) \in \Sigma$ we define the following differential operator

$$D_\lambda^n\, f^{(q)}(z) = D\,[\,D_\lambda^{n-1}\, f^{(q)}(z)]$$

$$= (-1)^q\, q!\, \frac{(\lambda-q-1)^n}{z^{q+1}} + \sum_{k=1}^{\infty} \frac{k!}{(k-q)!}\,(k+\lambda-q)^n\, a_k z^{k-q} \qquad \ldots(5.1.2)$$

$(q,n \in N_0)$ where $f^{(q)}(z)$ is the q^{th} derivative of $f(z)$ defined in (5.1.1) and

$$D[f(z)] \equiv \frac{(z^\lambda f(z))'}{z^{\lambda-1}} \qquad \ldots(5.1.3)$$

With help of the differential operator D_λ^n, we say that a function $f(z)$ belongs to Σ is in the class $A(n,q,\lambda,\alpha)$ if and only if

$$\mathrm{Re}\left\{-\frac{z\left(D_\lambda^n f^{(q)}(z)\right)'}{D_\lambda^n f^{(q)}(z)}\right\} > \alpha \qquad \ldots(5.1.4)$$

where $z \in U^*$; $\lambda \geq 0$; $0 \leq \alpha < 1+q$; $n,q \in N_0$ and D_λ^n is defined in (5.1.2).

MAIN RESULTS

We establish the following ten properties for a function $f(z)$ belonging to Σ in the class $A(n,q,\lambda,\alpha)$ defined under condition given in (5.1.4)

Coefficient Estimate

Theorem-1 : Let the function $f(z)$ defined by (5.1.1) be in the class Σ. Then the function $f(z)$ belong to the class $A(n,q,\lambda,\alpha)$ if and only if

$$\sum_{k=1}^{\infty} \frac{k!}{(k-q)!}(k+\lambda-q)^n (k+\alpha-q)a_k \leq (-1)^q\, q!\,(\lambda-q-1)^n (1+q-\alpha) \qquad \ldots(5.2.1)$$

$$(0 \leq \alpha < 1+q ; \lambda \geq 0 ; n,q \in N_0)$$

Proof: Let us suppose that the inequality (5.2.1) hold true. Then in view of condition given in (5.1.4) and $|z|<1$ we have

$$\left|\frac{z\left(D_\lambda^n f^{(q)}(z)\right)'}{D_\lambda^n f^{(q)}(z)} + (1+q)\right|$$

$$\leq \frac{\sum_{k=1}^{\infty} \frac{k!}{(k-q)!}(\lambda+k-q)^n (k+1)a_k}{(-1)^q(\lambda-q-1)^n q! + \sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n a_k}$$

$$\leq (1+q-\alpha)$$

Therefore the values of the functions

$$\Phi(z) = \frac{z\left(D_\lambda^n f^{(q)}(z)\right)'}{D_\lambda^n f^{(q)}(z)} \qquad \ldots(5.2.2)$$

lie in a circle which is a centered at $\omega = (1+q)$ and whose radius is $(1+q-\alpha)$.

Hence the function satisfies the condition given in (5.1.4).

Now conversely, assume that the function $f(z)$ is in the class $A(n,q,\lambda,\alpha)$. Then, we have

$$\text{Re}\left\{-\frac{z\left(D_\lambda^n f^{(q)}(z)\right)'}{D_\lambda^n f^{(q)}(z)}\right\}=$$

$$\text{Re}\left\{\frac{(-1)^q(\lambda-q-1)^n q!(q+1)-\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(k-q)a_k z^{k+1}}{(-1)^q(\lambda-q-1)^n q!+\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n a_k z^{k+1}}\right\}>\alpha$$

…(5.2.3)

for some $\alpha\,(0\le\alpha<1+q\,;\lambda\ge0\,;n,q\in N_0)$ and $z\in U^*$ choose value of z on the real axis so that $\Phi(z)$ given by (5.2.2) is real. Upon clearing the denominator in (5.2.3) and letting $z\to1^-$ through the real values we can see that inequality in (5.2.3) lead to inequality (5.2.1). It completes the proof of Theorem–1.

Theorem –2 : Let the function $f(z)$ defined by (5.1.1) be in the class $A(n,q,\lambda,\alpha)$. Then

$$a_k\le\frac{(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(k+\alpha-q)}\qquad\text{...(5.2.5)}$$

$(k\ge1\,;q,n\in N_0\,;\lambda\ge0)$

The result is sharp for the function $f(z)$ given by

$$f(z)=\frac{1}{z}+\frac{(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(k+\alpha-q)}z^k\qquad\text{…(5.2.6)}$$

$$(k\ge1;\ q,n\in N_0;\lambda\ge0)$$

Proof: As $f(z)\in A(n,q,\lambda,\alpha)$ therefore in (5.2.1) k^{th} term will be less then equal to the sum on L.H.S. of (5.2.1). Therefore (5.2.5) is true for the function defined in (5.2.6).

Distortion Theorem

Theorem-3 : If the function $f(z)$ defined by (5.1.1) is in the class $A(n,q,\lambda,\alpha)$ then

$$\left|f^{(m)}(z)\right|\ge$$

$$\left\{(-1)^m\frac{m!}{|z|}-\frac{(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)}{(1+\lambda-q)^n(1+\alpha-q)}|z|\right\}|z|^{-m}\qquad\text{…(5.2.7)}$$

and

$$\left|f^{(m)}(z)\right|\le\left\{(-1)^m\frac{m!}{|z|}+\frac{(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)}{(1+\lambda-q)^n(1+\alpha-q)}|z|\right\}|z|^{-m}$$

…(5.2.8)

$$(z\in U;0\le\alpha<(1+q)\,;q,n,m\in N_0;\lambda\ge0)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=\frac{1}{z}+\frac{(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)(1-q)!}{(1+\lambda-q)^n(1+\alpha-q)}z$$

…(5.2.9)

Proof: The function $f(z)$ is in the class $A(n,q,\lambda,\alpha)$ then in view of the assertion (5.2.1) of Theorem-1 we can see that

$$\frac{(1+\lambda-q)^n(1+\alpha-q)}{(1-q)!}\sum_{k=1}^{\infty}k!a_k$$

$$\le\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(k+\alpha-q)a_k$$

$$\le(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty}k!a_k\le\frac{(-1)^q q!(\lambda-q-1)^n(1+q-\alpha)(1-q)!}{(1+\lambda-q)^n(1+\alpha-q)}$$

…(5.2.10)

Now on differentiating both sides of (5.2.1) m times , we have

$$f^{(m)}(z)=(-1)^m\frac{m!}{z^{m+1}}+\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}a_k z^{k-m}$$

…(5.2.11)

Now taking the modulus of both sides of (5.2.11) and using (5.2.10) we at once arrive at the desired results in (5.2.7) and (5.2.8).

This completes the proof of Theorem-3.

Radii of Starlikeness and Convexity

Theorem – 4: Let the function defined by (5.1.1) be in the class $A(n,q,\lambda,\alpha)$. Then

(i) $f(z)$ is meromoriphically starlike of order $\Psi(0\le\Psi<1)$ in $|z|=r_1$, where

$$r_1=\inf_{k\ge1}\left\{\frac{k!(\lambda+k-q)^n(k+\alpha-q)}{(k-q)!(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)}\,\frac{(1-\Psi)}{(k+2-\Psi)}\right\}^{\frac{1}{k+1}}$$

…(5.2.12)

$(k\ge1\,;q,n\in N_0\,;\lambda\ge0)$

(ii) $f(z)$ is meromoriphically convex of order $\Psi(0\le\Psi<1)$ in $|z|<r_2$, where

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$$r_2 = \inf_{k\geq 1}\left\{\frac{k!(\lambda+k-q)^n(k+\alpha-q)}{(k-q)!(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)}\;\frac{(1-\Psi)}{k(k+2-\Psi)}\right\}^{\frac{1}{k+1}} \quad \ldots(5.2.13)$$

$(k\geq 1; q,n\in N_0; \lambda\geq 0)$

Each of these results is sharp for the function $f(z)$ given by (5.2.6)

Proof: Let $f(z)\in A(n,q,\lambda,\alpha)$. Then by Theorem-2 we have

$$a_k \leq \frac{(-1)^q(\lambda-q-1)^n(1+q-\alpha)q!}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)} \quad \ldots(5.2.14)$$

To obtain the radius of starlike function (5.1.1) given in (5.2.12) it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq (1-\Psi) \quad \ldots(5.2.15)$$

The L.H.S. of (5.2.15):

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum_{k=1}^{\infty}(k+1)a_k z^k}{\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k}\right| \leq \frac{\sum_{k=1}^{\infty}(k+1)a_k|z|^{k+1}}{1-\sum_{k=1}^{\infty}a_k|z|^{k+1}}$$

Then in view of (5.2.15) this will be bounded by ($1-\Psi$) therefore on

$$\frac{\sum_{k=1}^{\infty}(k+2-\Psi)a_k|z|^{k+1}}{(1-\Psi)} \leq 1 \quad \ldots(5.2.16)$$

In view of (5.2.14) it follows that the inequality in (5.2.16) is true if

$$\frac{(k+2-\Psi)}{(1-\Psi)}|z|^{k+1} \leq \frac{k!(\lambda+k-q)^n(k+\alpha-q)}{(k-q)!(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)} \quad \ldots(5.2.17)$$

Setting $|z| = r_1$ in (5.2.17) we get desired result in (5.2.12).

Similarly, to prove that $f(z)$ is meromorphically convex of order Ψ it is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq (1-\Psi)$$

$(k\geq 1; q,n\in N_0; \lambda\geq 0)$

for radius $|z| = r_2$ given in (5.2.13).

Closure Theorems

Theorem – 5 : Let the function $f_j(z)$ $(j=1,2,....)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty}a_{k,j}z^k \quad \text{for } z\in U \quad \ldots(5.2.18)$$

be in the class $A(n,q,\lambda,\alpha)$ for every $j=1,2,...,m$. Then the function $F(z)$ defined by

$$F(z) = \frac{1}{z} + \sum_{k=1}^{\infty}b_k z^k$$

belongs to the class $A(n,q,\lambda,\alpha)$ where

$$b_k = \frac{1}{m}\sum_{k=1}^{\infty}a_{k,j} \qquad (k\in N)$$

Proof: Since $f_j(z)\in A(n,q,\lambda,\alpha)$, it follows from Theorem-1 that

$$\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(k+\alpha-q)a_{k,j} \leq (-1)^q q!(\lambda-q-1)^n(1+q-\alpha) \quad \ldots(5.2.19)$$

for every $j=1,2,..,m$. Hence

$$\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)b_k$$
$$= \frac{1}{m}\sum_{j=1}^{m}\left[\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)a_{k,j}\right]$$
$$\leq (-1)^q q!(\lambda-q-1)^n(1+q-\alpha)$$

By Theorem – 1, it follows that $F(z)\in A(n,q,\lambda,\alpha)$.

Theorem –6: The class $A(n,q,\lambda,\alpha)$ is closed under convex linear combination.

Proof: Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.2.18) be in the class $A(n,q,\lambda,\alpha)$. It is sufficient to show that the function

$$H(z) = t\, f_1(z) + (1-t)f_2(z) \qquad (0\leq t\leq 1)$$

is also belongs to the class $A(n,q,\lambda,\alpha)$.

Since $(0\leq t\leq 1)$

$$H(z) = \frac{1}{z} + \sum_{k=1}^{\infty}[ta_{k,1}+(1-t)a_{k,2}]z^k$$

We observe that

$$\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)\{ta_{k,1}+(1-t)a_{k,2}\}$$
$$= t\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)\,a_{k,1}$$

$$+ (1-t)\,\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)\,a_{k,2}$$

$$\leq (-1)^q q!(\lambda-q-1)^n(1+q-\alpha)$$

By Theorem –1, it follows that $H(z)\in A(n,q,\lambda,\alpha)$.

Theorem – 7: Let $f_0(z) = \frac{1}{z}$ and

$$f_k(z)=\frac{1}{z}+\frac{(-1)^q(\lambda-q-1)^n(1+q-\alpha)q!}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)}z^k$$

for $k=1,2,\ldots$. Then $f(z)\in A(n,q,\lambda,\alpha)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z)$$

where $\mu_k \geq 0$ and $\sum_{k=0}^{\infty}\mu_k = 1$

Proof: Let $f(z) = \sum_{k=0}^{\infty}\mu_k f_k(z)$,

Then in view of $\sum_{k=0}^{\infty}\mu_k = 1$ we have

$$f(z) = \frac{1}{z}+\sum_{k=1}^{\infty}\mu_k\frac{(-1)^q(\lambda-q-1)^n\,(1+q-\alpha)q!}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)}z^k$$

Now we have

$$\sum_{k=1}^{\infty}\mu_k\frac{(-1)^q(\lambda-q-1)^n(1+q-\alpha)q!}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)}\,\frac{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)^n}{(-1)^q(\lambda-q-1)^n(1+q-\alpha)q!}$$

$$= \sum_{k=1}^{\infty}\mu_k = 1-\mu_0 \leq 1 \qquad \ldots(5.2.20)$$

Therefore from equation (5.2.20) we conclude in view of Theorem–1 that $f(z)\in A(n,q,\lambda,\alpha)$.

Conversely : As $f(z)\in A(n,q,\lambda,\alpha)$ then in view of Theorem–2 we have

$$a_k \leq \frac{(-1)^q(\lambda-q-1)^n\,(1+q-\alpha)q!}{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)}$$

$$(k\in N)$$

On setting

$$\mu_k = \frac{\frac{k!}{(k-q)!}(\lambda+k-q)^n(\alpha+k-q)}{(-1)^q(\lambda-q-1)^n\,(1+q-\alpha)q!}\,a_k\ ,$$

We have $\sum_{k=0}^{\infty}\mu_k = 1$

therefore it is true that

$$\mu_0 = 1-\sum_{k=1}^{\infty}\mu_k$$

Now, it follows that

$$f(z) = \sum_{k=0}^{\infty}\mu_k f_k(z)$$

This completes the proof of Theorem–7.

Integral Operator for function

Theorem – 8: Let the function $f(z)$ given by (5.1.1) be in $A(n,q,\lambda,\alpha)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)du \qquad \ldots(5.2.21)$$

$$(0<u\leq 1; 0<c<\infty)$$

is in $A(n,q,\lambda,\delta)$ where

$$\delta = \frac{(1+\alpha-q)(1+q)(c+2)-(1-q)(1+q-\alpha)c}{c(1+q-\alpha)+(c+2)(1+\alpha-q)} \qquad \ldots(5.2.22)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=\frac{1}{z}+\frac{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)}{\frac{1}{(1-q)!}(1+\lambda-q)^n(1+\alpha-q)}z \qquad \ldots(5.2.23)$$

Proof : Let $f(z)\in A(n,q,\lambda,\alpha)$. Then for

$$F(z) = c\int_0^1 u^c f(uz)du$$

we have

$$F(z) = \frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{c}{c+k+1}\right)a_k z^k$$

Since $f(z)\in A(n,q,\lambda,\alpha)$. We have from Theorem.-1

i.e. $$\frac{\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)a_k}{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)} \leq 1$$

Now for function $F(z)$ in view of Theorem –1, It is sufficient to show that the largest δ satisfies

$$\frac{\sum_{k=1}^{\infty}\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\delta-q)\left(\frac{c}{c+k+1}\right)a_k}{(-1)^q(\lambda-q-1)^n q!(1+q-\delta)} \leq 1$$

$$\ldots(5.2.24)$$

Therefore, the value of δ satisfies the range

$$\frac{(k+\delta-q)}{(1+q-\delta)}\left(\frac{c}{c+k+1}\right) \leq \frac{(k+\alpha-q)}{(1+q-\alpha)} \qquad \ldots(5.2.25)$$

for each $k\in N$. From (5.2.25), we obtain

$$\delta \leq \frac{(k+\alpha-q)(1+q)(c+k+1)-c(k-q)(1+q-\alpha)}{c(1+q-\alpha)+(c+k+1)(k+\alpha-q)} = H(k)$$

We see that $H(k)$ is increasing therefore $H(k) \geq H(1)$ it means $\delta = H(1) \leq H(k)$.

Now the result in (5.2.21) follows immediately.

Modified Hadamard Product

Theorem-9: Let the function $f_j(z)(j=1,2)$ defined by (5.2.18) be in the class $A(n,q,\lambda,\alpha)$. Then the modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k \quad \ldots(5.2.26)$$

$(f_1 * f_2)(z) \in A(n,q,\lambda,\delta)$ where

$$\delta = (1+q) - \frac{2(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)^2}{\frac{1}{(1-q)!}(1+\lambda-q)^n(1+\alpha-q)^2 + (-1)^q(\lambda-q-1)^n q!(1+q-\alpha)^2} \quad \ldots(5.2.27)$$

The result is sharp for the functions $f_j(z)\ (j=1,2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)}{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)} z^k \quad \ldots(5.2.28)$$

$(k \in N)$

Proof: In order to prove Theorem –9, we have to find the largest δ in view of Theorem –2 and (5.2.26) i.e.

$$\frac{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\delta-q)}{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)} a_{k,1} a_{k,2} \leq 1 \quad \ldots(5.2.29)$$

Since $f_1(z)$ and $f_2(z)$ are in $A(n,q,\lambda,\alpha)$ then for $f_1(z)$ and $f_2(z)$ we have the following inequalities

$$\sum_{k=1}^{\infty} \frac{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)}{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)} a_{k,1} \leq 1$$

and

$$\sum_{k=1}^{\infty} \frac{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)}{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)} a_{k,2} \leq 1$$

Now by Cauchy – Schwartz inequality and then in view of Theorem - 2, we have

$$\frac{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)}{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)} \sqrt{a_{k,1}.a_{k,2}} \leq 1 \quad \ldots(5.2.30)$$

Then for the convolution $f_1(z)$ and $f_2(z)$ in class $A(n,q,\lambda,\alpha)$ in view of (5.2.29) and (5.2.30) we have

$$\sqrt{a_{k,1}.a_{k,2}} \leq \frac{(k+\alpha-q)(1+q-\delta)}{(k+\delta-q)(1+q-\alpha)}$$

Now in the light of inequality (5.2.30) we have

$$\frac{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)}{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)} \leq \frac{(k+\alpha-q)(1+q-\delta)}{(k+\delta-q)(1+q-\alpha)} \quad ..(5.2.31)$$

On simplifying the inequality (5.2.31) we obtain

$\delta \leq G(k)$

where the function $G(k)$ is

$$G(k) = (1+q) - \frac{(-1)^q(\lambda-q-1)^n q!(1+q-\alpha)^2(k+1)}{\frac{k!}{(k-q)!}(k+\lambda-q)^n(k+\alpha-q)^2 + (-1)^q(\lambda-q-1)^n q!(1+q-\alpha)^2} \quad \ldots(5.2.32)$$

We see that $G(k)$ is an increasing function of k, therefore $G(1) \leq G(k)$

i.e. $\delta = G(1)$

which means in view of (5.2.32) at $k=1$ gives (5.2.27).

5.3 APPLICATIONS

As application of the theorems established in this section contain certain known and new results for the known class $\Sigma^*(\alpha)$ of univalent meromorphic functions with positive coefficients.

We illustrate some results deduced from our main theorems as follows:

(i) For the choice of $n=q=\lambda=0$ in Theorem-1, we get known corollary due to Kavitha et al [3, p.111].

(ii) If in Theorem-2 we put $n=q=\lambda=0$ then it reduces to the known corollary due to Kavitha et al [3, p.111].

(iii) For $n=q=\lambda=0$ and $m=0$ in Theorem-3, we get the following corollary:

Corollary-1. If $f(z) \in \Sigma^*(\alpha)$ then

$$\frac{1}{|z|} - \frac{1-\alpha}{1+\alpha} \leq |f(z)| \leq \frac{1}{|z|} + \frac{1-\alpha}{1+\alpha} \quad \ldots(5.3.1)$$

$(z \in U^*;\ 0 \leq \alpha < 1)$

The result is sharp for the function $f(z)$ given in (5.4.5).

(iv) At $n=q=\lambda=0$ in Theorem–4 provides the following corollary:

Corollary-2. Let the function $f(z)$ defined by (5.2.1) be in the class $\Sigma^*(\alpha)$.

Then $f(z)$ is meromorphically starlike of order $\Psi(0 \leq \Psi < 1)$ in $|z| < r_1$, where

$$r_1 = \operatorname*{Inf}_{k \geq 1} \left\{ \frac{(k+\alpha)}{(1-\alpha)} \frac{(1-\Psi)}{(k+2-\Psi)} \right\}^{\frac{1}{k+1}} \quad \ldots(5.3.2)$$

The result in (5.4.7) is sharp for the function $f(z)$ given by (5.4.3).

(v) If in Theorem-8 we take $n = q = \lambda = 0$ then it reduces to the know corollary due to Uralegaddi and Ganigi [6].

(vi) For $n = q = \lambda = 0$ in Theorem-9 we have the following corollary:

Corollary-3. Let the function $f_j(z)(j = 1,2)$ defined by (5.2.4) be in the class $\Sigma^*(\alpha)$. Then $(f_1 * f_2)(z) \in \Sigma^*(\delta)$, Where

$$\delta = 1 - \frac{2(1-\alpha)^2}{(1+\alpha)^2 + (1-\alpha)^2} \qquad \ldots(5.3.3)$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{1-\alpha}{k+\alpha} z^k \qquad \ldots(5.3.4)$$

$(k \in N)$

REFERENCES

[1] AOUF, M. K., On a certain class of meromorphic univalent functions with positive coefficients, Rend. Mat. Appl., 7 (11), No. 2, (1991), 209-219.

[2] ATSHAN, W, G., Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative, II. Surv. Math. Appl., 3, (2008), 67-77.

[3] KAVITHA, S., SIVASUBRAMANIAN, S. and MUTHUNAGAI, K., A new class of meromorphic function with positive coefficients, Bulletin of mathematical analysis and applications, 3, (2010), 109-121.

[4] MOGRA, M. L., REDDY, T. R. and JUNEJA, O. P., Meromorphic univalent functions with positive coefficients, Bull. Austral. Math. Soc., 32, No. 2, (1985), 161-176.

[5] SRIVASTAVA, H. M. and OWA, S., *Current Topics in Analytic Function Theory*, World Scientific Publishing Co., New Jersey, (1992).

[6] URALEGADDI, B. A. and GANIGI, M. D., A certain class of meromorphically starlike functions with positive coefficients, Pure Appl. Math. Sci., 26, (1987), 75-81.