On Extended Fractional Fourier Transform

A. S. Gudadhe# And Naveen A *

Govt. Vidarbha Institute of Science and Humanities, Amravati. (M. S.)
* Vidyabharti mahavidyalaya, Amravati. (M. S.)

Abstract: This paper studies extended fractional Fourier transform, which is generalization of fractional Fourier transform with two more parameters. Here we have illustrated some properties of kernel. Also we have proved some operational formulae. We also deal with parsevel's relation.

Key words: Extended fractional Fourier transform. Parseval's relation.

1. Introduction: The Fourier transform(FT) is a most popular transform in the theory of optics, signal processing and many other branches of engineering. The concept of FT of fractional ordered is introduced by Namias [5] in 1980 which is known as fractional Fourier transform (FrFT) given by

$$F^{\alpha}[f(t)](u) = \frac{e^{\left[i\left(\frac{\pi-2\alpha}{4}\right)\right]}}{\sqrt{2\pi sin\alpha}} \int_{-\infty}^{\infty} e^{-i\left[\frac{(t^2+u^2)}{2}cot\alpha - tucsc\alpha\right]} f(t)dt \qquad (1.1)$$

Due to the vast applicability of FrFT in signal filtering [6], optics [4], quantum mechanics[5], number of scientists extended the definition of FrFT in different manners. Shih C. in 1995 [8] used the integer ordered Fourier transform as basis and fractionalized Fourier transform in a different way. In 2008 Zhou et al [9] defined complex fractional Fourier transform and extended it. Pei, Liu and Lai [7] in 2012 had given extended versions of FrFT by using four generalizations of Hermite Gaussian function.

Lohmann [3] suggested a new definition,

$$F^{\alpha}[f(t)](u) = \int_{-\infty}^{\infty} e^{i\pi\frac{(t^2+u^2)}{\lambda f_s tan\,\alpha}} e^{\frac{-2i\pi tu}{\lambda f_s sin\,\alpha}} \; f(t)dt \qquad (1.2)$$

This definition ensures the involvement of the wavelength λ and focal length f as $f_s = f\ \sin\varphi$.

In the same paper [3], he had also given the definition,

$$F^{\alpha}_{a,b}[f(t)](u) = F^{\alpha}_{a,b}(u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt \qquad (1.3)$$

$$= \int_{-\infty}^{\infty} f(t) K_{a,b}^{\alpha}(t,u) dt$$

where $K_{a,b}^{\alpha}(t,u) = e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}$

Particularly when $a = b = \frac{1}{\sqrt{\lambda f_s}}$ it gives (1.2) where as $a = b = \sqrt{\frac{-1}{2\pi}}$ gives (1.1). In [2] we have enhanced the domain of extended FrFT to the spaces of generalized functions and obtained its inversion as,

$$f(t) = abcsc\alpha \int_{-\infty}^{\infty} F_{a,b}^{\alpha}(u) K_{a,b}^{-\alpha}(t,u) du \quad (1.4)$$

where $K_{a,b}^{-\alpha}(t,u) = e^{-i\pi(a^2t^2+b^2u^2)cot\alpha + 2i\pi abtucsc\alpha}$

Here we put forward some operation transform formulae concerning to this extended FrFT.

2. Operations on extended fractional Fourier transform:

In this section we proved properties on extended fractional Fourier transform.

2.1 Theorem (Time shift property): If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}[f(t+\tau)](u) = e^{i\pi[a^2\tau^2 cos\alpha sin\alpha + 2ab\tau usin\alpha]} F_{a,b}^{\alpha}(u + \frac{a}{b}\tau cos\alpha)$$

Proof: By (1.3)

$$F_{a,b}^{\alpha}[f(t+\tau)](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t+\tau) dt \quad (2.1)$$

Let we take $t + \tau = T$ then $t = T - \tau$ therefore $dt = dT$

Therefore equation (2.1) will be

$$F_{a,b}^{\alpha}[f(t+\tau)](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2(T-\tau)^2+b^2u^2)cot\alpha - 2ab(T-\tau)ucsc\alpha]} f(T) dT$$

$$= e^{i\pi a^2\tau^2 cot\alpha + 2i\pi ab\tau ucsc\alpha} \int_{-\infty}^{\infty} e^{i\pi[(a^2T^2+b^2u^2)cot\alpha - 2a^2\tau Tcot\alpha - 2abTucsc\alpha]} f(T) dT$$

$$= e^{i\pi a^2\tau^2 cot\alpha + 2i\pi ab\tau ucsc\alpha} \int_{-\infty}^{\infty} e^{i\pi[(a^2T^2+b^2u^2)cot\alpha - 2ab\left(u+\frac{a}{b}\tau cos\alpha\right)Tcsc\alpha]} f(T) dT$$

$$= e^{i\pi a^2\tau^2 cot\alpha + 2i\pi ab\tau u csc\alpha} \int_{-\infty}^{\infty} e^{i\pi[a^2T^2 cot\alpha - 2ab\left(u+\frac{a}{b}\tau cos\alpha\right)T csc\alpha]} e^{i\pi b^2(u+\frac{a}{b}\tau cos\alpha)^2 cot\alpha}$$

$$e^{-i\pi b^2(u+\frac{a}{b}\tau cos\alpha)^2 cot\alpha} e^{i\pi b^2u^2 cot\alpha} f(T)dT$$

$$= e^{i\pi a^2\tau^2 cot\alpha + 2i\pi ab\tau u csc\alpha} \int_{-\infty}^{\infty} e^{i\pi(a^2T^2 + b^2(u+\frac{a}{b}\tau cos\alpha)^2)cot\alpha - 2i\pi ab\left(u+\frac{a}{b}\tau cos\alpha\right)T csc\alpha}$$

$$e^{-i\pi b^2(\frac{a}{b}\tau cos\alpha)^2 cot\alpha - 2i\pi b^2 u\frac{a}{b}\tau cos\alpha cot\alpha} f(T)dT \qquad (2.2)$$

Here we can write extended fractional Fourier transform with parameter $\left(u+\frac{a}{b}\tau cos\alpha\right)$ as

$$F_{a,b}^{\alpha}[f(t+\tau)]\left(u+\frac{a}{b}\tau cos\alpha\right) = \int_{-\infty}^{\infty} e^{i\pi(a^2t^2 + b^2(u+\frac{a}{b}\tau cos\alpha)^2)cot\alpha - 2i\pi ab\left(u+\frac{a}{b}\tau cos\alpha\right)t csc\alpha} f(t)dt$$

Therefore (2.2) will be

$$F_{a,b}^{\alpha}[f(t+\tau)](u) = e^{i\pi a^2\tau^2 cot\alpha + 2i\pi ab\tau u csc\alpha}\ e^{-i\pi b^2(\frac{a}{b}\tau cos\alpha)^2 cot\alpha - 2i\pi abu\tau cos\alpha cot\alpha}$$

$$F_{a,b}^{\alpha}[f(t)]\left(u+\frac{a}{b}\tau cos\alpha\right)$$

$$= e^{i\pi a^2\tau^2 cot\alpha\,(1-cos^2\alpha) + 2i\pi ab\tau u csc\alpha\,(1-cos^2\alpha)} F_{a,b}^{\alpha}[f(t)]\left(u+\frac{a}{b}\tau cos\alpha\right)$$

Therefore

$$F_{a,b}^{\alpha}[f(t+\tau)](u) = e^{i\pi a^2\tau^2 cos\alpha sin\alpha + 2i\pi ab\tau u sin\alpha}\ F_{a,b}^{\alpha}[f(t)]\left(u+\frac{a}{b}\tau cos\alpha\right)$$

2.2 Theorem: If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}\left[f(t)e^{2i\pi vt}\right](u) = e^{i\pi[\frac{2b}{a}uv cos\alpha - \frac{v^2}{a^2}sin\alpha cos\alpha]} F_{a,b}^{\alpha}\left(u-\frac{v sin\alpha}{ab}\right)$$

Proof: As

$$F_{a,b}^{\alpha}\left[f(t)e^{2i\pi vt}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtu csc\alpha]} f(t)e^{2i\pi vt}\,dt$$

$$= e^{i\pi b^2u^2 cot\alpha} \int_{-\infty}^{\infty} e^{i\pi a^2t^2 cot\alpha - 2i\pi abt\left(u-\frac{v sin\alpha}{ab}\right)csc\alpha} f(t)dt$$

$$= e^{i\pi b^2u^2 cot\alpha} \int_{-\infty}^{\infty} e^{i\pi a^2t^2 cot\alpha + i\pi b^2\left(u-\frac{v sin\alpha}{ab}\right)^2 cot\alpha - 2i\pi abt\left(u-\frac{v sin\alpha}{ab}\right)csc\alpha} e^{-i\pi b^2\left(u-\frac{v sin\alpha}{ab}\right)^2 cot\alpha} f(t)dt$$

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$$= e^{-i\pi b^2\left(\frac{vsin\alpha}{ab}\right)^2 cot\alpha + 2i\pi b^2 \frac{uvsin\alpha}{ab} cot\alpha} \int_{-\infty}^{\infty} e^{i\pi\left[a^2t^2+b^2\left(u-\frac{vsin\alpha}{ab}\right)^2\right]cot\alpha - 2i\pi abt\left(u-\frac{vsin\alpha}{ab}\right)csc\alpha} f(t)dt$$

Therefore

$$F_{a,b}^{\alpha}[f(t)e^{2i\pi vt}](u) = e^{i\pi[\frac{2b}{a}uvcos\alpha - \frac{v^2}{a^2}sin\alpha cos\alpha]} F_{a,b}^{\alpha}[f(t)]\left(u - \frac{vsin\alpha}{ab}\right)$$

2.3 Theorem: If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}[f'(t)](u) = 2i\pi abusin\alpha F_{a,b}^{\alpha}(u) + \frac{a}{b}cos\alpha\, F_{a,b}^{'\alpha}(u)$$

Proof: As

$$F_{a,b}^{\alpha}[f'(t)](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f'(t)dt$$

$$= -2i\pi a^2 cot\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\, t.f(t)dt + 2i\pi abucsc\alpha F_{a,b}^{\alpha}(u) \quad (2.3)$$

Differentiating (1.3) with respect to u we get

$$F_{a,b}^{'\alpha}[f(t)](u) = F_{a,b}^{'\alpha}(u)$$

$$= \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}[2i\pi b^2ucot\alpha - 2i\pi abtcsc\alpha]f(t)dt$$

$$\Rightarrow 2i\pi abcsc\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\, t.f(t)dt$$

$$= 2i\pi b^2ucot\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt - F_{a,b}^{'\alpha}(u)$$

Therefore

$$\int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\, t.f(t)dt = \frac{2i\pi b^2ucot\alpha F_{a,b}^{\alpha}(u) - F_{a,b}^{'\alpha}(u)}{2i\pi abcsc\alpha} \quad (2.4)$$

Applying (2.4) in (2.3) then

$$F_{a,b}^{\alpha}[f'(t)](u) = -2i\pi a^2cot\alpha \frac{2i\pi b^2ucot\alpha F_{a,b}^{\alpha}(u) - F_{a,b}^{'\alpha}(u)}{2i\pi abcsc\alpha} + 2i\pi abucsc\alpha F_{a,b}^{\alpha}(u)$$

$$= 2i\pi abucsc\alpha F_{a,b}^{\alpha}(u)[-cos^2\alpha + 1] + \frac{a}{b}cos\alpha F_{a,b}^{'\alpha}(u)$$

Therefore

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$$F_{a,b}^{\alpha}[f'(t)](u) = \frac{a}{b} cos\alpha F'^{\alpha}_{a,b}(u) + 2i\pi abusin\alpha F^{\alpha}_{a,b}(u) \qquad (2.5)$$

2.4 Theorem: If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}\left[\frac{d^n}{dt^n} f(t)\right](u) = \left[2i\pi abusin\alpha + \frac{a}{b} cos\alpha \frac{d}{du}\right]^n F_{a,b}^{\alpha}[f(t)](u)$$

Proof: By (2.5)

$$F_{a,b}^{\alpha}[f'(t)](u) = \frac{a}{b} cos\alpha F'^{\alpha}_{a,b}(u) + 2i\pi abusin\alpha F^{\alpha}_{a,b}(u)$$

Let we take II[nd] order derivative extended FrFT will be

$$F_{a,b}^{\alpha}[f''(t)](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f''(t)dt$$

$$= e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f'(t)\Big|_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\, 2i\pi[a^2tcot\alpha - abucsc\alpha]f'(t)dt$$

$$= (2i\pi a^2cot\alpha)^2 \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\, t^2 f(t)dt$$

$$-2(4i^2\pi^2a^3bucot\alpha csc\alpha) \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\, tf(t)dt$$

$$+[2i\pi a^2cot\alpha + (2i\pi abucsc\alpha)^2] \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt \qquad (2.6)$$

By (1.3) differentiation of extended FrFT with respect to u will be

$$\frac{d}{du} F_{a,b}^{\alpha}[f(t)](u) = F'^{\alpha}_{a,b}(u) = 2i\pi b^2cot\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} uf(t)dt$$

$$-2i\pi abtcsc\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt$$

Differentiating one more time with respect to u then

$$\frac{d^2}{du^2} F_{a,b}^{\alpha}[f(t)](u) = F''^{\alpha}_{a,b}(u) = (2i\pi b^2cot\alpha)^2 \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt$$

$$-2(2i\pi)^2 ab^3 ucot\alpha csc\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} t\, f(t)dt$$

$$+2i\pi b^2 cot\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} f(t)dt$$

$$+(2i\pi abcsc\alpha)^2 \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} t^2 f(t)dt$$

Therefore

$$\int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} t^2 f(t)dt$$

$$= \frac{1}{(2i\pi abcsc\alpha)^2}\Bigg[\frac{d^2}{du^2}F_{a,b}^{\alpha}(u)$$

$$-(2i\pi b^2 cot\alpha)^2 \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} f(t)dt$$

$$+2(2i\pi)^2 ab^3 ucot\alpha csc\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} t\, f(t)dt$$

$$-2i\pi b^2 cot\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]} f(t)dt\Bigg] \qquad (2.7)$$

Therefore by using (2.4) and (2.7) in (2.6) we get

$$F_{a,b}^{\alpha}[f''(t)](u)$$

$$= \frac{(2i\pi a^2 cot\alpha)^2}{(2i\pi abcsc\alpha)^2}\Bigg[F_{a,b}^{''\alpha}(u) - (2i\pi b^2 ucot\alpha)^2 F_{a,b}^{\alpha}(u) - 2i\pi b^2 cot\alpha F_{a,b}^{\alpha}(u)$$

$$+2(2i\pi)^2 ab^3 ucot\alpha csc\alpha \frac{2i\pi b^2 ucot\alpha F_{a,b}^{\alpha}(u) - F_{a,b}^{'\alpha}(u)}{2i\pi abcsc\alpha}\Bigg]$$

$$-2(2i\pi)^2 a^3 bucot\alpha csc\alpha \frac{2i\pi b^2 ucot\alpha F_{a,b}^{\alpha}(u) - F_{a,b}^{'\alpha}(u)}{2i\pi abcsc\alpha}$$

$$+[2i\pi a^2 cot\alpha + (2i\pi abucsc\alpha)^2]F_{a,b}^{\alpha}(u)$$

$$= [(2i\pi abu)^2 \sin^2\alpha + 2i\pi a^2 cos\alpha sin\alpha]F_{a,b}^{\alpha}(u)$$

$$+2(2i\pi)a^2 ucos\alpha sin\alpha \frac{d}{du}F_{a,b}^{\alpha}(u) + \frac{a^2 cos^2\alpha}{b^2}\frac{d^2}{du^2}F_{a,b}^{\alpha}(u)$$

Therefore

$$F_{a,b}^{\alpha}[f''(t)](u) = \left[2i\pi abusin\alpha + \frac{a}{b}cos\alpha\frac{d}{du}\right]^2 F_{a,b}^{\alpha}(u) \tag{2.8}$$

From (2.5) and (2.8) we get the n^th^ order derivative as

$$F_{a,b}^{\alpha}\left[\frac{d^n}{dt^n}f(t)\right](u) = \left[2i\pi abusin\alpha + \frac{a}{b}cos\alpha\frac{d}{du}\right]^n F_{a,b}^{\alpha}(u)$$

Hence proved.

2.5 Theorem: If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}[t^n f(t)](u) = \left[\frac{b}{a}ucos\alpha - \frac{sin\alpha}{2i\pi ab}\frac{d}{du}\right]^n F_{a,b}^{\alpha}(u)$$

Proof: By (2.4)

$$F_{a,b}^{\alpha}[tf(t)](u) = \left[\frac{b}{a}ucos\alpha - \frac{sin\alpha}{2i\pi ab}\frac{d}{du}\right] F_{a,b}^{\alpha}[f(t)](u) \tag{2.9}$$

Also from (2.7)

$$\begin{aligned} F_{a,b}^{\alpha}[t^2 f(t)](u) \\ &= \frac{1}{(2i\pi abcsc\alpha)^2}\Big[F''^{\alpha}_{a,b}(u) - [2i\pi b^2 cot\alpha + (2i\pi b^2 ucot\alpha)^2]F_{a,b}^{\alpha}(u) \\ &\quad + 2(2i\pi)^2 ab^3 ucos\alpha csc^2\alpha F_{a,b}^{\alpha}[tf(t)](u)\Big] \\ &= \frac{b^2u^2cos^2\alpha}{a^2}F_{a,b}^{\alpha}(u) - \frac{ucos\alpha sin\alpha}{i\pi a^2}F'^{\alpha}_{a,b}(u) \\ &\qquad - \frac{cos\alpha sin\alpha}{2i\pi a^2}F_{a,b}^{\alpha}(u) + \frac{sin^2\alpha}{(2i\pi ab)^2}F''^{\alpha}_{a,b}(u) \\ &= \left[\frac{b}{a}ucos\alpha - \frac{sin\alpha}{2i\pi ab}\frac{d}{du}\right]^2 F_{a,b}^{\alpha}(u) \end{aligned} \tag{2.10}$$

Similarly $F_{a,b}^{\alpha}[t^3 f(t)](u) = \left[\frac{b}{a}ucos\alpha - \frac{sin\alpha}{2i\pi ab}\frac{d}{du}\right]^3 F_{a,b}^{\alpha}(u)$ (2.11)

From (2.9), (2.10) and (2.11) general form of product by an integral variable extended FrFT will be

$$F_{a,b}^{\alpha}[t^n f(t)](u) = \left[\frac{b}{a}ucos\alpha - \frac{sin\alpha}{2i\pi ab}\frac{d}{du}\right]^n F_{a,b}^{\alpha}(u)$$

2.6 Theorem (Primitive property): If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

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$$F_{a,b}^{\alpha}\left[\iint f(t)\right](u) = e^{-i\pi b^2u^2tan\alpha}\,\frac{b}{a}sec\alpha\int F_{a,b}^{\alpha}[f(t)](u)\,e^{i\pi b^2u^2tan\alpha}\,du + ce^{-i\pi b^2u^2tan\alpha}$$

If $\alpha - \frac{\pi}{2}$ is not multiplied π

Proof: We have derivative property of the extended fractional Fourier transform as

$$F_{a,b}^{\alpha}[f'(t)](u) = 2i\pi abusin\alpha F_{a,b}^{\alpha}[f(t)](u) + \frac{a}{b}cos\alpha F'^{\alpha}_{a,b}[f(t)](u)$$

Let us assume that $g'(t)$ as $f(t)$, that is $g'(t) = f(t)$ therefore $g(t) = \int f(t)\,dt$

Therefore from derivative property we get the differential equation as

$$F_{a,b}^{\alpha}[g'(t)](u) = 2i\pi abusin\alpha F_{a,b}^{\alpha}[g(t)](u) + \frac{a}{b}cos\alpha F'^{\alpha}_{a,b}[g(t)](u)$$

Consider this as Ist order differential equation, its solution will be

$$F_{a,b}^{\alpha}[g(t)](u) = e^{-\int 2i\pi b^2utan\alpha du}\,\left[\iint \frac{b}{a}sec\alpha F_{a,b}^{\alpha}[g'(t)](u)\,e^{\int 2i\pi b^2utan\alpha du}\,du + c\right]$$

$$F_{a,b}^{\alpha}\left[\iint f(t)\right](u) = e^{-i\pi b^2u^2tan\alpha}\,\frac{b}{a}sec\alpha\int F_{a,b}^{\alpha}[f(t)](u)\,e^{i\pi b^2u^2tan\alpha}\,du + ce^{-i\pi b^2u^2tan\alpha}$$

If $\alpha - \frac{\pi}{2}$ is not multiplied π

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2.7 Theorem (Time inversion): If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}[f(-t)](u) = F_{a,b}^{\alpha}(-u)$$

Proof: Being very simple proof is omitted.

2.8 Theorem (Parity): If $F_{a,b}^{\alpha}[f(t)](u)$ is extended fractional Fourier transform of $f(t)$ then

$$F_{a,b}^{\alpha}[-f(-t)](u) = -F_{a,b}^{\alpha}(-u)$$

Proof: Proof is simple hence omitted.

3. Parseval relation property: $F_{a,b}^{\alpha}[f(t)](u)$ is the extended fractional Fourier transform then

$$\int_{-\infty}^{\infty} f(t)\,g^*(t)dt = \; abcsc\alpha \int_{-\infty}^{\infty} F_{a,b}^{\alpha}(u)G_{a,b}^{*\alpha}(u)\,du$$

Proof: As (1.3) let we take an extended FrFT of function $g(t)$ as

$$F_{a,b}^{\alpha}[g(t)](u) = G_{a,b}^{\alpha}(u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]}\,g(t)dt$$

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Inversion of $G_{a,b}^{\alpha}(u)$ will be

$$g(t) = abcsc\alpha \int_{-\infty}^{\infty} e^{-i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} G_{a,b}^{\alpha}(u)\, du$$

Then conjugation of above $g(t)$ is

$$g^*(t) = abcsc\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} G_{a,b}^{*\alpha}(u)\, du$$

Therefore

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)dt = \int_{-\infty}^{\infty} f(t) \left\{ abcsc\alpha \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} G_{a,b}^{*\alpha}(u)\, du \right\} dt$$

$$= abcsc\alpha \int_{-\infty}^{\infty} G_{a,b}^{*\alpha}(u) \left\{ \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt \right\} du$$

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)dt = abcsc\alpha \int_{-\infty}^{\infty} F_{a,b}^{\alpha}(u) G_{a,b}^{*\alpha}(u)\, du$$

Table of extended fractional Fourier properties:

Si. No.	$f(t)$	Extended fractional Fourier transform
1	$f(t)$	$\int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha]} f(t)dt$
2	$f(t+\tau)$	$e^{i\pi[a^2\tau^2 cos\alpha sin\alpha + 2ab\tau u sin\alpha]} F_{a,b}^{\alpha}[f(t)](u + \frac{a}{b}\tau cos\alpha)$
3	$f(t)e^{2i\pi vt}$	$e^{i\pi[\frac{2b}{a}uv cos\alpha - \frac{v^2}{a^2} sin\alpha cos\alpha]} F_{a,b}^{\alpha}[f(t)]\left(u - \frac{v sin\alpha}{ab}\right)$
4	$f'(t)$	$2i\pi abu sin\alpha F_{a,b}^{\alpha}(u) + \frac{a}{b} cos\alpha F'^{\alpha}_{a,b}(u)$
5	$\frac{d^n}{dt^n} f(t)$	$\left[2i\pi abu sin\alpha + \frac{a}{b} cos\alpha \frac{d}{du}\right]^n F_{a,b}^{\alpha}(u)$
6	$t^n f(t)$	$\left[\frac{b}{a} u cos\alpha - \frac{sin\alpha}{2i\pi ab} \frac{d}{du}\right]^n F_{a,b}^{\alpha}(u)$
7	$\int f(t)\, dt$	$\frac{b}{a} sec\alpha\, e^{-i\pi b^2u^2 tan\alpha} \int e^{i\pi b^2u^2 tan\alpha} F_{a,b}^{\alpha}(u)du + ce^{-i\pi b^2u^2 tan\alpha}$

8	$f(-t)$	$F_{a,b}^{\alpha}(-u)$

Conclusion: In this paper we have established several operation transform formulae for the extended fractional Fourier transform which will be useful for application of the transform to solve partial differential equation. Moreover Parseval's identity is also obtained. Also note that similar results for fractional Fourier transform are special cases of these results with $a = b = 1$.

References:

[1] Almeida L. B. (1993): An Introduction to the Angular Fourier transform, IEEE.

[2] Gudadhe A. S. and Naveen A.(2013): Extended fractional Fourier transform of distributions of compact support. Communicated.

[3] Lohmann A. W. (1993): Image rotation, Wigner rotation, and the fractional Fourier transform, J. Opt. Soc. Am. A 10, 2181-2186.

[4] Mendlovic D. and Ozaktas H. M. (1993): Fractional Fourier transform and their optical implementation. I. J. Opt. Soc. Am. A 10,1875-1881.

[5] Namias V. (1980): The fractional order Fourier transform and its application to quantum mechanics, J. Inst. Math. Appl. 25, 241-265.

[6] Ozaktas H. M., Zulevsky Z. and Kutay M. A. (2001): The fractional Fourier transform with Applications in Optics and Signal Process, John Wiley and Sons Chichester.

[7] Pei, Lin and Lai (2012): The generalized fractional Fourier transform, ICASSP IEEE.

[8] Shih C. (1995): Fractionalization of Fourier transform, Optics communications, 118, 495-498.

[9] Zhou N., Hu L. and Fan H. (2008): The complex fractional Fourier transform, the complex Wigner transform and the entangled Wigner operator in EPR entangled state representation, Phys. Sce. 78, 1-6.

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