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On Indicatrices of Null Cartan Curves in $\mathbf{R_1^4}$

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Abstract

In this paper, we investigate indicatrices of null Cartan curves in Minkowski 4-space which lie on lightcone and pseudosphere, and give some characterizations for these curves to be a generalized helix in terms of Cartan curvatures.

1. Introduction

There are three types curve in a (M,g) Semi-Riemannian manifolds according to its velocity vectors: spacelike, timelike and (null) lightlike. As non-null curves (spacelike and timelike) have some similarities with Riemannian case, but some diffuculties arise for null curves.

Contrary to Riemannian case, general Frenet frame and its general Frenet equations are not unique as they depend on the parameter of the curve and the screen vector bundle. To deal this non-uniqueness problem, Bonnor [1] was the first who introduced a unique Frenet frame along null curves in R_1^4 with the minimum number of curvature functions. Bonnor's work was recently generalized by Ferrandez-Gimenez-Lucas [3] for null Cartan curves in M_1^{m+2} and they studied some classification theorem on null helices.

Recently physical importance of null Cartan curves has shown by Ferrandez-Gimenez-Lucas [4]. After that Çöken-Çiftçi [2] have followed the 3-dimensional notion of Bertrand curves and proved a characterization theorem for null helices in R_1^4. Also Sakaki [7] has shown a correspondence between the evolute of a null curve and the involute of a certain spacelike curve in the 4-dimensional Minkowski spacetime.

In this paper, we study relationships between a null Cartan curve and its indicatrices and give some characterizations.

2. Null Cartan Curves

The Minkowski space-time R_1^4 is the Euclidean 4-space equipped with the the indefinite flat metric given by $\langle,\rangle=-dx_1^2+dx_2^2+dx_3^2+dx_4^2$, where (x_1,x_2,x_3,x_4) is rectangular coordinate system of R_1^4. Recall that an arbitrary vector $v\in R_1^4\setminus\{0\}$ can be spacelike, timelike or null(lightlike), if $\langle v,v\rangle>0$, $\langle v,v\rangle<0$ or $\langle v,v\rangle=0$ and $v\neq 0$ respectively. In particular, the vector v=0 is spacelike. The norm of a vector v is given by $\|v\|=\sqrt{|\langle v,v\rangle|}$, and two vectors v and w are said to be orthogonal, if $\langle v,w\rangle=0$. An arbitrary curve $\alpha(t)$ in R_1^4, can be locally spacelike, timelike or null(lightlike), if all its velocity vectors $\alpha'(t)$ are spacelike, timelike or null, respectively. A spacelike or timelike curve $\alpha(s)$ is said to be parametrized by a pseudo-arclength parameter s, i.e. $\langle\alpha'(s),\alpha'(s)\rangle=\pm1$.

Let $\alpha(s)$ be a spacelike curve in R_1^4, parametrized by arclenght function s. Similar to the Euclidean case, a spacelike helix can be defined according to the tangent indicatrix: A spacelike curve in R_1^4 is said to be a general helix if its unit tangent indicatrix is contained in a hyperplane.

Recall that the pseudo-sphere, and the lightcone are hyperquadratics in R_1^4, respectively defined by

$$S_1^3(m,r)=\{x\in R_1^4:\langle x-m,x-m\rangle=r^2\}$$

and

$$\Lambda_0^3(m)=\{x\in R_1^4:\langle x-m,x-m\rangle=0\}$$

where r>0 is the radius and $m\in R_1^4$ is the center of hyperquadratics. In particular, for r=1 and m=0 we denote this hyperquadratics S_1^3 and Λ_0^3 respectively.

Now let $\gamma(t)$ be a null curve in R_1^4. Since $\langle\gamma'(t),\gamma'(t)\rangle=0$, classical methods for non-null curves does not work in this case. So we need a new construction for this curve. We say that a null curve $\gamma(t)$ in R_1^4 is parametrized by pseudo-arc if $\langle\gamma''(t),\gamma''(t)\rangle=1$. If a null curve $\gamma(t)$ in R_1^4 satisfies $\langle\gamma''(t),\gamma''(t)\rangle\neq0$, then $\langle\gamma''(t),\gamma''(t)\rangle>0$ and

$$u(t)=\int_{t_0}^{t} < \gamma''(t), \gamma''(t) > dt$$

becomes the pseudo-arc parameter.

Let us say that a null curve $\gamma(t)$ in R_1^4 with $<\gamma''(t),\gamma''(t)>\neq 0$ is a Cartan curve if $\{\gamma'(t),\gamma''(t),\gamma'''(t),\gamma^{(4)}(t)\}$ is linearly indipendent for any t.

For a null Cartan curve $\gamma(t)$ in R_1^4 with pseudo-arc parameter t, there exist a unique Frenet Frame $\{L,N,W_1,W_2\}$ such that

$$\gamma' = L,\ L'=W_1,\ N'=\kappa W_1+\tau W_2 \qquad 2.1$$
$$W_1' = -\kappa L-N,\quad W_2'=-\tau L$$

where

$$<L,L>=<N,N>=0 \qquad 2.2$$

$$<W_1,W_1>=<W_2,W_2>=1 \qquad 2.3$$

$$<L,W_i>=<N, W_i>=0\ ,\ <L,N>=1, i=1,2 \qquad 2.4$$

and $\{L,N,W_1,W_2\}$ and $\{\gamma',\gamma'',\gamma''',\gamma^{(4)}\}$ have the same orientation and $\{L,N,W_1,W_2\}$ is positively oriented. Also the functions κ and τ are called the Cartan curvatures of γ.

Definition 2.1 A null Cartan curve $\gamma:I\rightarrow R_1^4$ is said to be a generalized helix if there exists a constant vector $v\neq 0$ such that the product $\lambda=<L(t),v>\neq 0$ is constant.

Theorem 2.2 Let γ be a null Cartan curve. Then γ is a generalized helix if and only if its Cartan curvatures satisfy the following differential equation

$$(\kappa')^2=\tau^2(2\kappa+c) \quad ,\kappa'\neq 0 \qquad 2.5$$

where c is a constant.

The axis of generalized helix γ, in above theorem, is

$$v=-\lambda(\kappa L+N-\frac{\kappa'}{\tau}W_2) \qquad 2.6$$

where $\lambda=\sqrt{c}$ is a non-zero constant [See 4].

Teorem 2.3 Let γ be a Cartan curve in R_1^4. Then γ is a pseudo-spherical curve if and only if τ is a non-zero constant.

Teorem 2.4 Let γ be a Cartan curve in R_1^4. Then γ is a three-dimensional null helix if and only if there exists a fixed direction u such that

$$<L,u>=a \text{ and } <N,u>=b$$

where a and b are non-zero constants and $\{L,N,W_1,W_2\}$ is the Cartan frame of γ.

3. Conic Indicatrices of Null Cartan Curves

Let γ be a null Cartan curve in R_1^4. From (2.1) and (2.2), its indicatrix curves (L) and (N) are spacelike curves which lie on null cone. Then the unit tangential vector fields of these curves are

$$T_{(L)}=W_1 \qquad 3.1$$

and

$$T_{(N)} = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\kappa W_1+\tau W_2). \qquad 3.2$$

respectively. Therefore we can give following results:

Proposition 3.1 Let γ be a null Cartan curve with non-zero constant curvature κ. Then its conic indicatrix (L) is a spacelike generalized helix if and only if Cartan curvatures of γ satisfy following differential equation

$$\left(\left(\frac{1}{\tau}\right)'\right)^2 = \left(\frac{1}{8\kappa^3}\right)(4\kappa^2 - A\tau^2) \qquad 3.3$$

where $A=(\varepsilon/\mu)-1$.

Proof Suppose that (L) is a spacelike generalized helix in R_1^4, i.e. there exists a constant vector u such that the product $<T_{(L)},u>$ is a non-zero constant. Then we have from (3.1)

$$<W_1,u>=\mu. \qquad 3.4$$

Let

$$u=fL+gN+\mu W_1+hW_2$$

where μ is a non-zero constant and f,g and h are differentiable functions of the parameter t. Then

$$\left(\frac{du}{dt}\right) = (f - \mu\kappa - ht)L + (g' - \mu)N$$
$$+(f + g\kappa)W_1 + (h' + g\tau)W_2 \qquad . \qquad 3.5$$

As u is a constant vector we get

$$f=-2\mu\kappa^2((1/\tau))((1/\tau))', g=2\mu\kappa((1/\tau))((1/\tau))', h=-2\mu\kappa((1/\tau)).$$

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Thus the axis of the (L) is

$$u=-2\mu\kappa^2((1/\tau))((1/\tau))'L+2\mu\kappa((1/\tau))((1/\tau))'N+\mu W_1-2\mu\kappa((1/\tau))W_2. \quad 3.6$$

On the other side, as $<u,u>=\pm1=\varepsilon$ we obtain that equation (3.3).

Conversely, suppose the curvatures of γ satisfy the equation (3.3) and consider vector field along γ defined by (3.6) where $\mu=\sqrt{|(\varepsilon/(A+1))|}$.Then u is a constant vector and $<T_{(L)},u>=\mu$.

Proposition 3.2 Let γ be a generalized helix in R_1^4. Then its conic indicatrix (N) is a generalized spacelike helix which has same axis of the γ if and only if Cartan curvatures of γ satisfy the following differential equations

$$(\kappa')^2=A(\kappa^2+\tau^2)$$

where A is $((\mu^2)/(\lambda^2))$.

Proof Let assume that (N) be a generalized spacelike helix and (2.6) is its axis. As

$$<T_{(N)},v>=\mu\neq0,$$

we obtain that

$$\mu=<((-1)/(\sqrt{(\kappa^2+\tau^2)}))(\kappa W_1+\tau W_2),-\lambda(\kappa L+N-((\kappa')/\tau)W_2)> =((-\lambda\kappa')/(\sqrt{(\kappa^2+\tau^2)})).$$

Then we have

$$(\kappa')^2=A(\kappa^2+\tau^2) \quad 3.7$$

where A is $((\mu^2)/(\lambda^2))$.

Conversely, using the equation (3.7)

$$<T_{(N)},v>^2=((\lambda^2(\kappa')^2)/(\kappa^2+\tau^2)),$$

then we obtain that

$$<T_{(N)},v>^2=\lambda^2A$$

is non-zero constant.

4. Pseudospherical Indicatrices of null Cartan Curves

Let γ be a null Cartan curve in R_1^4. Again from (2.1) and (2.3), its indicatrix curves (W_1) and (W_2) is spacelike curve with $\kappa>0$ and lightlike curve respectively, which lie on pseudo-sphere. Then the unit tangential vector fields of these curves are

$$T_{(W_1)}=\frac{-1}{\sqrt{2\kappa}}(\kappa L+N)$$

and

$$T_{(W_2)}=-\tau L$$

respectively.

Thus we can give following results.

Proposition 4.1 Let γ be a Cartan curve in R_1^4 with κ is non-zero constant. Then γ is a three-dimensional null helix if and only if its pseudo-spherical indicatrix (W_1) is a generalized helix.

Proof Since γ is a three-dimensional null helix, there exists a fixed direction u such that $<L,u>=a$ and $<N,u>=b$, where a and b are non zero constants. Then direct computation shows that

$$<T_{(W_1)},u>=\frac{1}{\sqrt{2\kappa}}(\kappa<L,u>+<N,u>)$$

is non-zero constant.

Conversely, assume that (W_1) is a generalized helix, i.e. there exits a constant vector u such that

$$<T_{(W_1)},u>=\mu\neq0.$$

Then we get

$$\frac{1}{\sqrt{2\kappa}}(\kappa<L,u>+<N,u>)=\mu.$$

If we set

$$<L,u>=f \quad 4.1$$

where f is a differentiable function of t, we have

$$<N,u>=\mu\sqrt{2\kappa}-f\kappa. \quad 4.2$$

Then taking derivatives of (4.1) and (4.2) with respect to t, we have

$$<W_1,u>=f' \quad 4.3$$

and

$$\tau<W_2,u>=0. \quad 4.4$$

Now let asume that,

$$<W_2,u>=0. \quad 4.5$$

If we take derivative of (4.5) and account that (4.1), we have

$$-\tau f=0$$

so f=0.Then from (4.2) the product $\langle N,u\rangle$ is a non-zero constant. But there is no constant vector such that $\langle N,u\rangle$=cons.[5]. Thus $\tau=0$. Then γ is a 3-dimensional helix.

Proposition 4.2 Let γ be a generalized helix in R_1^4. Then its pseudo-spherical indicatrix (W_2) is a generalized helix which has same axis of the γ if and only if γ is pseudo-spherical curve.

Proof Let (W_2) be a generalized helix and consider its axis in above. Since

$$\langle T_{(W_2)},v\rangle=\mu\neq0,$$

we obtain that

$$\mu=\langle -\tau L,-\lambda(\kappa L+N-((\kappa')/\tau)W_2)\rangle=\lambda\tau.$$

Thus τ is a non-zero constant. This means that from Theorem 2.3 γ is a pseudo-spherical curve.

Conversely, as τ is a non-zero constant the product

$$\langle T_{(W_2)},v\rangle=\lambda\tau$$

is also non-zero constant.

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