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on Novel Approximation of (P,Q) - Trapezoidal and Midpoint Inequalities and Their Implementation in Post-quantum Calculus

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ABSTRACT:
In this work, we will show several improvements of Hermite-Hadamard inequalities for convex functions using the concept of post-quantum calculus. We will also demonstrate how the recent discoveries relate to the previous findings. Furthermore, we will present some applications of recently discovered inequalities along with several mathematical illustrations.

***KEYWORDS*: (*p*, q)-Hermite-Hadamard Inequality; (p, *q*)-derivative; (*p*, *q*)-integral; post-quantum calculus; convex function.**

INTRODUCTION:

The field of q-analysis, which Euler introduced in response to the increasing need for mathematical models that may explain quantum computing, has been the subject of several contemporary investigations. Due to its versatility, Q-calculus has become a key tool in the domains of quantum theory, mechanics, relativity, and number theory and also in combinatorics, fundamental hypergeometric functions, and orthogonal polynomials[1], [2].

Early scholars were the first to recognize the importance of convexity in a number of areas, therefore the idea of convexity has a long history. Overall, q-calculus is a powerful tool that has found applications in a wide range of fields. Its ability to describe non-commuting operators and its development of q-analogues of mathematical tools have made it a useful tool in quantum mechanics, computer science, and statistics, among other fields. As research into quantum computing continues to grow, it is likely that q-calculus will play an increasingly important role in the development of new algorithms and computational techniques[3].

One of the benefits of postquantum calculus is that it can be used to address issues that q-calculus cannot. In specific quantum systems, for instance, the behavior of a system may be affected by both the location and momentum of a particle. These systems may be modelled using the (p, q)-calculus.[4]

Post quantum calculus is an extension of q-calculus, which is a generalization of the traditional calculus that allows for calculations with q-numbers.

Q-numbers are a mathematical construct that generalizes the concept of integers and can be used to represent quantum systems[5]. Q-calculus has been applied to problems in physics, finance, and computer science, among other fields. Inequalities are crucial for addressing mathematical issues, but they also have useful uses in a variety of real-world contexts, including data analysis, function optimization, and setting conditions for certain outcomes. They are a crucial subject in any mathematics curriculum because they provide a basis for more complex mathematical ideas and concepts.

A mathematical function that meets the convexity property is said to be a convex function[6]. In plain English, a function is said to be convex if the line segment joining any two points on its graph sits wholly above the graph. In other words, the function seems "bowl-shaped" or "curves upward" when viewed from above.

If function $u(x)$ is convex on interval I, if any two points x_1 and x_2 in I and any γ where 0 <γ<1

$$u[\gamma x_1 + (1-\gamma)x_2] \leq \gamma u(x_1) + (1-\gamma)u(x_2).$$

The Hermite-Hadamard inequality is an important finding in mathematical analysis that shows a strong correlation among the average values of convex functions. This discrepancy was first studied in the late 19th century and is named after Charles Hermite and Jacques Hadamard[7][8]. By history, construction and uses of the Hermite-Hadamard inequality emphasizing its importance across a range of mathematical specialties and related disciplines. The Hermite Hadamard inequality of (p, q) – Calculus is that:

Let u: $[\alpha_1, \alpha_2] \to$ R be a convex differentiable function on $[\alpha_1, \alpha_2]$ and 0 < q < p ≤ 1. Then we have

$$u\left(\frac{q\alpha_1 + p\alpha_2}{p+q}\right) \leq \frac{1}{p(\alpha_2 - \alpha_1)} \int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x \leq \frac{u(q\alpha_1) + u(p\alpha_2)}{p+q}. \quad (4)$$

Similarly, the utilizing the right (p, q)-integral, demonstrated Hermite-Hadamard inequality for convex mappings, then

$$u\left(\frac{p\alpha_1 + q\alpha_2}{p+q}\right) \leq \frac{1}{p(\alpha_2 - \alpha_1)} \int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u(x)^{\alpha_2} d_{p,q}x \leq \frac{u(p\alpha_1) + u(q\alpha_2)}{p+q}. \quad (5)$$

The main objectives of research is that the use (p, q)-calculus approaches, establish various extensions of Hermite-Hadamard for q and (p, q) function. The establish connection between the new result and the previous finding results.

PRELIMINARIES

In this section we discuss some definition of quantum calculus and also implement on it.

Definition no 1: The q-number, which is represented by the quantum number notation $[n]_q$, is defined for any positive integer n.

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + q^3 + \cdots + q^{n-1}, \quad (2.1)$$

Where the degree of the polynomial is n-1 in relation to the deformation component q, which can take on either real or complex numbers.

Definition no 2: The (p, q) numbers is introduced by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \quad (2.2)$$

Where $[n]_{p,q}$ is symmetric then $[n]_{p,q} = [n]_{q,p}$ and if $p = 1$then the (p, q) numbers are reduced into q-numbers and if $q = 1$ then reduced back into ordinary numbers n[9].

It means that the (p, q) numbers of Post Quantum calculus are performing same role as q-numbers of Quantum calculus.

Definition no 3:[10], [11] Let u:$[\alpha_1, \alpha_2]$→ℝ be continuous. The left q-derivative of u at x ∈$[\alpha_1, \alpha_2]$ is

$${}_{\alpha_1}D_q u(x) = \frac{u(x) - u(qx + (1-q)\alpha_1)}{(1-q)(x-\alpha_1)}, \quad x \neq \alpha_1 \quad (2.3)$$

If $\alpha_1 = 0$ $and {}_{\alpha_1}D_q u(x) = D_q u(x)$then (1) become

$$D_q u(x) = \frac{u(x) - u(qx)}{(1-q)x}, \quad x \neq 0. \quad (2.4)$$

This equation is Similar to q- Jackson derivative formula.

And the right q-derivative is

$${}^{\alpha_2}D_q u(x) = \frac{u(qx + (1-q)\alpha_1) - u(x)}{(1-q)(\alpha_2 - x)}, \quad x \neq \alpha_2 \quad (2.5)$$

Definition no 4: [12] Let u:$[\alpha_1, \alpha_2]$→ℝ and it is continuous on$[\alpha_1, \alpha_2]$. The q-integral of u at x ∈$[\alpha_1, \alpha_2]$is

$$\int_{\alpha_1}^{\alpha_2} u(x)^{\alpha_2} d_q x = (1-q)(\alpha_2 - \alpha_1) \sum_{n=o}^{\infty} q^n u(\alpha_2 q^n + (1-q^n)\alpha_1). \quad (2.6)$$

If α_1=0 then equation (2) becomes

$$\int_0^{\alpha_2} u(x)^{\alpha_2} d_q x = (1-q)(\alpha_2) \sum_{n=o}^{\infty} q^n u(\alpha_2). \tag{2.7}$$

The equation (4) is similar to q-Jackson Integral.

And the right q-integral is

$$\int_{\alpha_1}^{\alpha_2} u(x)^{\alpha_2} d_q x = (1-q)(\alpha_2-\alpha_1) \sum_{n=o}^{\infty} q^n u(\alpha_1 q^n + (1-q^n)\alpha_2). \tag{2.8}$$

Definition no 5: [13] Let $u: [\alpha_1, \alpha_2\ ] \to \mathbb{R}$ is continuous function, and $(p,q)_{\alpha_1}$-derivative of u at $x\ \in [\alpha_1, \alpha_2]$is define as:

$${}_{\alpha_1}D_{p,q}u(x) = \frac{u(px+(1-p)\alpha_1) - u(qx+(1-q)\alpha_1)}{(p-q)(x-\alpha_1)} \tag{2.9},$$

With $0 < q < p \leq 1$.

Definition no 6: Similarly, $(p,q)^{\alpha_2}$-derivative of u at $x\ \in [\alpha_1, \alpha_2] \to \mathbb{R}$ is define as:

$${}^{\alpha_2}D_{p,q}u(x) = \frac{u(qx+(1-q)\alpha_2) - u(px+(1-p)\alpha_2)}{(p-q)(x-\alpha_2)} \tag{2.10}$$

Definition no 7: Let $u: [\alpha_1, \alpha_2\ ] \to \mathbb{R}$ is continuous function, and $(p,q)_{\alpha_1}$-integral of u at $x\ \in [\alpha_1, \alpha_2]$is define as:

$$\int_{\alpha_1}^{x} u(t)_{\alpha_1} d_{p,q} t = (p-q)(x-\alpha_1) \sum_{n=o}^{\infty} \frac{q^n}{p^{n+1}} u\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right)\alpha_1\right), \tag{2.11}$$

With $0 < q < p \leq 1$.

Definition no 8: [14] Similarly, $(p,q)^{\alpha_2}$-Integral of u at $x\ \in [\alpha_1, \alpha_2]$is define as:

$$\int_{x}^{\alpha_2} u(t)^{\alpha_2} d_q t = (p-q)(\alpha_2-x) \sum_{n=o}^{\infty} \frac{q^n}{p^{n+1}} u\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right)\alpha_2\right), \tag{2.12}$$

With $0 < q < p \leq 1$.

Lemma 1: If a function $u: [\alpha_1, \alpha_2] \to \mathbb{R}$ is Convex $\alpha_1 < \alpha_2$, then the following inequality hold for $(p,q)^{\alpha_2}$-Integral:

$$u\left(\frac{p\alpha_1 + q\alpha_2}{[2]_{p,q}}\right) \leq \frac{1}{p(\alpha_2-\alpha_1)} \int_{p\alpha_{1+(1-p)\alpha_2}}^{\alpha_2} u(x)^{\alpha_2} d_{p,q} x \leq \frac{pu(\alpha_1) + qu(\alpha_2)}{[2]_{p,q}}, \tag{2.13}$$

Where $0 < q < p \leq 1$

Lemma 2: If a function $u: [\alpha_1, \alpha_2] \to \mathbb{R}$ is Convex $\alpha_1 < \alpha_2$, then following inequality satisfied for $(p,q)_{\alpha_1}$-Integral:

$$u\left(\frac{q\alpha_1 + p\alpha_2}{[2]_{p,q}}\right) \leq \frac{1}{p(\alpha_2-\alpha_1)} \int_{\alpha_1}^{p\alpha_1+(1-p)\alpha_2} u(x)_{\alpha_1} d_{p,q} x \leq \frac{qu(\alpha_1) + pu(\alpha_2)}{[2]_{p,q}}, \tag{2.14}$$

Where $0 < q < p \leq 1$

3. New Trapezoidal type inequalities for (p, q)-integrals

Lemma 3: If u: $[\alpha_1,\alpha_2]\rightarrow\mathbb{R}$ is p, q-differentiable function that way$[_{\alpha_1}D_{p,q}u]$ and $[^{\alpha_2}D_{p,q}u]$ are continuous and integrable on $[\alpha_1,\alpha_2]$, then

$$\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2}u\,(x)^{\alpha_2}d_{p,q}x\right\}$$

$$=\frac{\alpha_2-\alpha_1}{2}\left[\int_0^1 qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}\,t+\int_0^1 qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}\,t\right]$$

Proof:

$$I_1=\int_0^1 qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}t$$

$$=\frac{q}{(\alpha_2-\alpha_1)(p-q)}\int_0^1 u(pt\alpha_2+(1-pt)\alpha_1)-u(qt\alpha_2+(1-qt)\alpha_1)d_{p,q}t$$

$$=\frac{q}{(\alpha_2-\alpha_1)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}u\left(\frac{q^n}{p^n}\alpha_2+\left(1-\frac{q^n}{p^n}\right)\alpha_1\right)-\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}u\left(\frac{q^{n+1}}{p^{n+1}}\alpha_2+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\alpha_1\right)\right]$$

$$=\frac{q}{(\alpha_2-\alpha_1)}\left[\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}u\left(\frac{q^n}{p^n}\alpha_2+\left(1-\frac{q^n}{p^n}\right)\alpha_1\right)-\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}u\left(\frac{q^{n+1}}{p^{n+1}}\alpha_2+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\alpha_1\right)\right]$$

$$=\frac{q}{(\alpha_2-\alpha_1)}\left[\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\,u\left(\frac{q^n}{p^n}\,\alpha_2+\left(1-\frac{q^n}{p^n}\right)\alpha_1\right)-\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}\,u\left(\frac{q^{n+1}}{p^{n+1}}\alpha_2+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\alpha_1\right)+\frac{1}{q}u(\alpha_2)\right]$$

$$=\frac{q}{(\alpha_2-\alpha_1)}\left[\left(\frac{1}{p}-\frac{1}{q}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^n}\,u\,\left(\frac{q^n}{p^n}\alpha_2+\left(1-\frac{q^n}{p^n}\right)\alpha_1\right)+\frac{1}{q}u(\alpha_2)\right]$$

$$=\frac{1}{(\alpha_2-\alpha_1)}\left[\frac{-1}{p(\alpha_2-\alpha_1)}\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+u(\alpha_2)\right]$$

$$(\alpha_2-\alpha_1)I_1=\left[\frac{-1}{p(\alpha_2-\alpha_1)}\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+u(\alpha_2)\right]\tag{1}$$

Similarly,

$$I_2=\int_0^1 qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t$$

$$=\frac{q}{(\alpha_2-\alpha_1)(p-q)}\int_0^1 u(pt\alpha_1+(1-pt)\alpha_2)-u(qt\alpha_1+(1-qt)\alpha_2)d_{p,q}t$$

$$=\frac{q}{(\alpha_2-\alpha_1)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}u\left(\frac{q^{n+1}}{p^{n+1}}\alpha_1+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\alpha_2\right)-\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}u\left(\frac{q^n}{p^n}\alpha_1+\left(1-\frac{q^n}{p^n}\right)\alpha_2\right)\right]$$

$$=\frac{q}{(\alpha_2-\alpha_1)}\left[\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}u\left(\frac{q^{n+1}}{p^{n+1}}\alpha_1+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\alpha_2\right)-\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\,u\left(\frac{q^n}{p^n}\alpha_1+\left(1-\frac{q^n}{p^n}\right)\alpha_2\right)\right]$$

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$$= \frac{q}{(\alpha_2-\alpha_1)}\left[\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}u\left(\frac{q^{n+1}}{p^{n+1}}\alpha_2+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\alpha_1\right)+\frac{1}{q}u(\alpha_1)-\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}u\left(\frac{q^n}{p^n}\alpha_1+\left(1-\frac{q^n}{p^n}\right)\alpha_2\right)\right]$$

$$= \frac{q}{(\alpha_2-\alpha_1)}\left[\left(\frac{1}{q}-\frac{1}{p}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^n}\,u\left(\frac{q^n}{p^n}\,\alpha_2+\left(1-\frac{q^n}{p^n}\right)\alpha_1\right)+\frac{1}{q}\,u(\alpha_2)\right]$$

$$= \frac{1}{(\alpha_2-\alpha_1)}\left[\frac{1}{p(\alpha_2-\alpha_1)}\int_{p\alpha_1(1-p)\alpha_2}^{\alpha_2}u(x)^{\alpha_2}d_{p,q}x+u(\alpha_1)\right]$$

$$(\alpha_2-\alpha_1)I_2 = \left[\frac{1}{p(\alpha_2-\alpha_1)}\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+u(\alpha_1)\right] \tag{2}$$

By equation 1 and 2 we get

$$\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2}u\,(x)^{\alpha_2}d_{p,q}x\right\}$$

$$= \frac{\alpha_2-\alpha_1}{2}\left[\int_0^1 qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}t+\int_0^1 qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t\right]$$

Hence proved.

Theorem 1: If $|_{\alpha_1}D_q u|$ and $|^{\alpha_2}D_q u|$are convex, then the following inequality holds according to Lemma (3) hypotheses:

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2}u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$\leq \frac{q(\alpha_2-\alpha_1)}{2[3]_{p,q}}\left[|_{\alpha_1}D_{p,q}u(\alpha_2)|+|^{\alpha_2}D_{p,q}u(\alpha_1)|+\frac{([3]_{p,q}-[2]_{p,q})\left(|_{\alpha_1}D_{p,q}u(\alpha_1)|+|^{\alpha_2}D_{p,q}u(\alpha_2)|\right)}{[2]_{p,q}}\right]$$

Proof: we know by lemma 3 and Taking mode both sides

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2}u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$= \frac{\alpha_2-\alpha_1}{2}\left[\left|\int_0^1 qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}t\right|+\left|\int_0^1 qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t\right|\right]$$

By using convexity of $|_{\alpha_1}D_{p,q}u|$ and $|^{\alpha_2}D_{p,q}u|$

$$\leq \frac{\alpha_2-\alpha_1}{2}\left[\int_0^1 qt\{t|_{\alpha_1}D_{p,q}u(\alpha_2)|+(1-t)|_{\alpha_1}D_{p,q}u(\alpha_1)|d_{p,q}t\}+\int_0^1 qt\{t|^{\alpha_2}D_{p,q}u(\alpha_1)|+(1-t)|^{\alpha_2}D_{p,q}u(\alpha_2)|\}d_{p,q}t\right]$$

$$= \frac{\alpha_2-\alpha_1}{2}\Bigg[|_{\alpha_1}D_{p,q}u(\alpha_2)|\int_0^1 qt^2d_{p,q}t+|_{\alpha_1}D_{p,q}u(\alpha_1)|\int_0^1 qt(1-t)qtd_{p,q}t+|^{\alpha_2}D_{p,q}u(\alpha_1)|\int_0^1 qt^2d_{p,q}t$$
$$+|^{\alpha_2}D_{p,q}u(\alpha_2)|\int_0^1 qt(1-t)qtd_{p,q}t\Bigg]$$

$$= \frac{\alpha_2-\alpha_1}{2}\left[|_{\alpha_1}D_{p,q}u(\alpha_2)|\frac{q}{[3]_{p,q}}+|_{\alpha_1}D_{p,q}u(\alpha_1)|\frac{[2]_{p,q}-[3]_{p,q}}{[2]_{p,q}[3]_{p,q}}+|^{\alpha_2}D_{p,q}u(\alpha_1)|\frac{q}{[3]_{p,q}}+|^{\alpha_2}D_{p,q}u(\alpha_2)|\frac{[2]_{p,q}-[3]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right]$$

$$= \frac{\alpha_2-\alpha_1}{2}\left[\left\{|_{\alpha_1}D_{p,q}u(\alpha_2)|+|^{\alpha_2}D_{p,q}u(\alpha_1)|\right\}\frac{q}{[3]_{p,q}}+\left\{|_{\alpha_1}D_{p,q}u(\alpha_2)|+|^{\alpha_2}D_{p,q}u(\alpha_1)|\right\}\frac{q([2]_{p,q}-[3]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right]$$

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Then, we get

$$\leq \frac{q(\alpha_2-\alpha_1)}{2[3]_{p,q}}\left[|_{\alpha_1}D_{p,q}u(\alpha_2)|+|^{\alpha_2}D_{p,q}u(\alpha_1)|+\frac{([3]_{p,q}-[2]_{p,q})\left(|_{\alpha_1}D_{p,q}u(\alpha_1)|+|^{\alpha_2}D_{p,q}u(\alpha_2)|\right)}{[2]_{p,q}}\right]$$

Hence proved.

Theorem 2: If,$|_{\alpha_1}D_{p,q}u|^{\gamma}$,$|^{\alpha_2}D_{p,q}u|^{\gamma}$ *and* $\gamma \geq 1$and it is convex function, then inequality holds according to Lemma (3) hypotheses:

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2}u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$\leq \frac{q(\alpha_2-\alpha_1)}{2[2]_{p,q}}\left[\left(\frac{[2]_{p,q}|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}+([3]_{p,q}-[2]_{p,q})|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}}{[3]_{p,q}}\right)^{\frac{1}{\gamma}}\right.$$
$$\left.+\left(\frac{[2]_{p,q}|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}+([3]_{p,q}-[2]_{p,q})|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}}{[3]_{p,q}}\right)^{\frac{1}{\gamma}}\right]$$

Proof: We know by lemma 3 and theorem 1 give

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1}u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2}u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$=\frac{\alpha_2-\alpha_1}{2}\left[\left|\int_0^1 qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}t\right|+\left|\int_0^1 qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t\right|\right]$$

By using power mean inequality

$$\leq \frac{\alpha_2-\alpha_1}{2}\left[\left(\int_0^1 qtd_{p,q}t\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 qt|_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)|d_{p,q}t\right)^{\frac{1}{\gamma}}\right.$$
$$\left.+\left(\int_0^1 qtd_{p,q}t\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 qt|^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)|d_{p,q}t\right)^{\frac{1}{\gamma}}\right]$$

By the convexity of $|_{\alpha_1}D_{p,q}u|^{\gamma}$ and $|^{\alpha_2}D_{p,q}u|^{\gamma}$, we have

$$\leq \frac{\alpha_2-\alpha_1}{2}\left[\left(\int_0^1 qtd_{p,q}t\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 qt\{t|_{\alpha_1}D_{p,q}u(\alpha_2)|+(1-t)|_{\alpha_1}D_{p,q}u(\alpha_1)|\}d_{p,q}t\right)^{\frac{1}{\gamma}}\right.$$
$$\left.+\left(\int_0^1 qtd_{p,q}t\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 qt\{t|^{\alpha_2}D_{p,q}u(\alpha_1)|+(1-t)|^{\alpha_2}D_{p,q}u(\alpha_2)|\}d_{p,q}t\right)^{\frac{1}{\gamma}}\right]$$

By taking integration we get

$$\leq \frac{q(\alpha_2-\alpha_1)}{2[2]_{p,q}}\left[\left(\frac{[2]_{p,q}|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}+([3]_{p,q}-[2]_{p,q})|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}}{[3]_{p,q}}\right)^{\frac{1}{\gamma}}\right.$$
$$\left.+\left(\frac{[2]_{p,q}|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}+([3]_{p,q}-[2]_{p,q})|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}}{[3]_{p,q}}\right)^{\frac{1}{\gamma}}\right]$$

Hence proved.

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Theorem 3: If $|_{\alpha_1}D_q u|^{\gamma}|^{\alpha_2}D_q u|^{\gamma}$ *and* $\gamma > 1$and it is convex, then inequality holds according to Lemma (3) hypotheses:

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$=\frac{q(\alpha_2-\alpha_1)}{2}\left(\frac{1}{[\gamma+1]_{p,q}}\right)^{\frac{1}{\gamma}}\left[\left(\frac{|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\delta}+(p+q-1)|_{\alpha_1}D_{p,q}u(\alpha_1)|^{\delta}}{[2]_{p,q}}\right)^{\frac{1}{\delta}}\right.$$
$$\left.+\left(\frac{|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\delta}+(p+q-1)|^{\alpha_2}D_{p,q}u(\alpha_2)|}{[2]_{p,q}}\right)^{\frac{1}{\delta}}\right]$$

Where $\gamma^{-1}+\delta^{-1}=1$

Proof: The lemma 1 and theorem 1 give

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$=\frac{\alpha_2-\alpha_1}{2}\left|\left[\int_0^1 qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}t+\int_0^1 qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t\right]\right|$$

$$\leq\frac{\alpha_2-\alpha_1}{2}\left[\int_0^1 qt|_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)|d_{p,q}t+\int_0^1 qt|^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)|d_{p,q}t\right]$$

By Holder inequality and lemma 1, we get

$$\leq\frac{\alpha_2-\alpha_1}{2}\left[\left(\int_0^1(qt)^{\gamma}d_qt\right)^{\frac{1}{\gamma}}\left(\int_0^1|_{\alpha_1}D_qu(t\alpha_2+(1-t)\alpha_1)|^{\delta}d_qt\right)^{\frac{1}{\delta}}\right.$$
$$\left.+\left(\int_0^1(qt)^{\gamma}d_qt\right)^{\frac{1}{\gamma}}\left(\int_0^1|^{\alpha_2}D_qu(t\alpha_1+(1-t)\alpha_2)|^{\delta}d_qt\right)^{\frac{1}{\delta}}\right]$$

By the convexity of $|_{\alpha_1}D_{p,q}u|^{\gamma}$ and $|^{\alpha_2}D_{p,q}u|^{\gamma}$, we have

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2}d_{p,q}x\right\}\right|$$

$$\leq\frac{\alpha_2-\alpha_1}{2}\left[\left(\int_0^1(qt)^{\gamma}d_{p,q}t\right)^{\frac{1}{\gamma}}\left(\int_0^1\{t|_{\alpha_1}D_{p,q}u(\alpha_2)|+(1-t)|_{\alpha_1}D_{p,q}u(\alpha_1)|\}d_{p,q}t\right)^{\frac{1}{\delta}}\right.$$
$$\left.+\left(\int_0^1(qt)^{\gamma}d_{p,q}t\right)^{\frac{1}{\gamma}}\left(\int_0^1\{t|^{\alpha_2}D_{p,q}u(\alpha_1)|+(1-t)|^{\alpha_2}D_{p,q}u(\alpha_2)|\}d_{p,q}t\right)^{\frac{1}{\delta}}\right]$$

By taking integration we get

$$=\frac{q(\alpha_2-\alpha_1)}{2}\left(\frac{1}{[\gamma+1]_{p,q}}\right)^{\frac{1}{\gamma}}\left[\left(\frac{|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\delta}+(p+q-1)|_{\alpha_1}D_{p,q}u(\alpha_1)|^{\delta}}{[2]_{p,q}}\right)^{\frac{1}{\delta}}\right.$$
$$\left.+\left(\frac{|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\delta}+(p+q-1)|^{\alpha_2}D_{p,q}u(\alpha_2)|}{[2]_{p,q}}\right)^{\frac{1}{\delta}}\right]$$

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Thus, the proof is completed.

4. New Midpoint type inequalities for (p, q)-integrals

Lemma 4: If u: $[[\alpha_1, \alpha_2] \subset$ R is p, q-differentiable function that way$[{}_{\alpha_1}D_{p,q}u]$ and $[{}^{\alpha_2}D_{p,q}u]$ are continuous and integrable on$[\alpha_1, \alpha_2]$, then

$$u\left(\frac{\alpha_2+\alpha_1}{2}\right)-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2}d_{p,q}x\right\}$$

$$=\frac{\alpha_2-\alpha_1}{2}\left[\int_0^{\frac{1}{2}} qt_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}+\int_{\frac{1}{2}}^{1}(qt-1)_{\alpha_1}D_{p,q}u(t\alpha_2+(1-t)\alpha_1)d_{p,q}t\right.$$

$$\left.+\int_0^{\frac{1}{2}}-qt^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t+\int_{\frac{1}{2}}^{1}(1-qt)^{\alpha_2}D_{p,q}u(t\alpha_1+(1-t)\alpha_2)d_{p,q}t\right]$$

Proof: It is simple to demonstrate by following the technique described in Lemma 3

Theorem 4: If $|{}_{\alpha_1}D_{p,q}u|$ and $|{}^{\alpha_2}D_{p,q}u|$are convex, then the following inequality holds according to Lemma (4) hypotheses:

$$\left|u\left(\frac{\alpha_2+\alpha_1}{2}\right)-\frac{1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2}d_{p,q}x\right]\right|$$

$$\leq\frac{\alpha_2-\alpha_1}{2}\left[|{}_{\alpha_1}D_{p,q}u(\alpha_2)|\frac{4p^2q-3p^2-3pq-3q^2}{4([3]_{p,q}[2]_{p,q})}+|{}_{\alpha_1}D_{p,q}u(\alpha_1)|\frac{2(2q^3-2p^3+2p^2q+2pq^2+3p^2}{8([3]_{p,q}[2]_{p,q})}\right.$$
$$\left.+|{}^{\alpha_2}D_{p,q}u(\alpha_1)|\frac{4p^2q-3p^2-3pq-3q^2}{4([3]_{p,q}[2]_{p,q})}+|{}^{\alpha_2}D_{p,q}u(\alpha_2)|\frac{2(2q^3-2p^3+2p^2q+2pq^2+3p^2}{8([3]_{p,q}[2]_{p,q})}\right]$$

Proof: It is simple to demonstrate by following the technique described in theorem 1

Theorem 5: If$|{}_{\alpha_1}D_{p,q}u|^{\gamma}$, $|{}^{\alpha_2}D_{p,q}u|^{\gamma} and\ \gamma \geq 1$and it is convex function, then the inequality holds according to Lemma (4) hypotheses:

$$\left|u\left(\frac{\alpha_2+\alpha_1}{2}\right)-\frac{1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1}d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2}d_{p,q}x\right]\right|$$

$$\leq\frac{\alpha_2-\alpha_1}{2}\left[\left(\frac{q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}}\times\left(|{}_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}\frac{q}{8[3]_{p,q}}+|{}_{\alpha_1}D_{p,q}u(\alpha_1)|^{\gamma}\frac{[3]_{p,q}+q^2}{8([3]_{p,q}[2]_{p,q})}\right)^{\frac{1}{\gamma}}\right.$$

$$+\left(\frac{2-q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}}\times\left(|{}_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}\frac{6-q[2]_{p,q}}{8([3]_{p,q}[2]_{p,q})}+|{}_{\alpha_1}D_{p,q}u(\alpha_1)|^{\gamma}\frac{5q-2q^2-2}{8[3]_{p,q}}\right)^{\frac{1}{\gamma}}$$

$$+\left(\frac{q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}}\times\left(|{}^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}\frac{q}{8[3]_{p,q}}+|{}^{\alpha_2}D_{p,q}u(\alpha_2)|^{\gamma}\frac{[3]_{p,q}+q^2}{8([4]_{p,q}+q[2]_{p,q})}\right)^{\frac{1}{\gamma}}$$

$$\left.+\left(\frac{2-q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}}\times\left(|{}^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}\frac{6-q[2]_{p,q}}{8([3]_{p,q}[2]_{p,q})}+|{}^{\alpha_2}D_{p,q}u(\alpha_2)|^{\gamma}\frac{5q-2q^2-2}{8[3]_{p,q}}\right)^{\frac{1}{\gamma}}\right]$$

Proof: It is simple to demonstrate by following the technique described in theorem 2.

Theorem 6: If$|{}_{\alpha_1}D_{p,q}u|^{\gamma}$ $|{}^{\alpha_2}D_{p,q}u|^{\gamma} and\ \gamma > 1$and it is convex function, then the inequality holds by using Lemma (4) hypotheses:

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$$\left|u\left(\frac{\alpha_2+\alpha_1}{2}\right)-\frac{1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2} d_{p,q}x\right]\right|$$

$$\leq \frac{\alpha_2-\alpha_1}{2}\left[q\left(\frac{1}{2^{\gamma+1}[\gamma+1]_q}\right)^{\frac{1}{\gamma}} \times\left(|_{\alpha_1}D_q u(\alpha_2)|\frac{1}{4[2]_{p,q}}+|_{\alpha_1}D_q u(\alpha_1)|\frac{1+2q}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}|qt-1|^{\gamma} d_q t\right)^{\frac{1}{\gamma}} \times\left(|_{\alpha_1}D_q u(\alpha_2)|\frac{3}{4[2]_{p,q}}+|_{\alpha_1}D_q u(\alpha_1)|\frac{6q-1}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}$$

$$+\left(\frac{1}{2^{\gamma+1}[\gamma+1]_q}\right)^{\frac{1}{\gamma}} \times\left(|^{\alpha_2}D_q u(\alpha_1)|\frac{1}{4[2]_{p,q}}+|^{\alpha_2}D_q u(\alpha_2)|\frac{1+2q}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}|1-qt|^{\gamma} d_q t\right)^{\frac{1}{\gamma}} \times\left(|^{\alpha_2}D_q u(\alpha_1)|\frac{3}{4[2]_{p,q}}+|^{\alpha_2}D_q u(\alpha_2)|\frac{6q-1}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}\right]$$

Where $\gamma^{-1}+\delta^{-1}=1$

Proof: It is simple to demonstrate by following the technique described in theorem 3

5. Examples of (P, Q)-Hermite-Hadamard inequality

This section Discuss the effectiveness of the newly build inequalities by giving examples and discuss these it useful like basic Hermite-Hadamard inequality.

Example 1: let $u=x^2+2$ is convex function on [0, 1] with $p=q=\frac{1}{2}$ then the L.H.S of inequality is that

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2} d_{p,q}x\right\}\right|$$

$$=\left|\frac{2+3}{2}-\frac{1}{2(1-0)}\left\{\int_0^{\frac{1}{2}}(x^2+2)_{\alpha_1} d_{p,q}x+\int_{\frac{1}{2}}^{1}(x^2+2)^{\alpha_2} d_{p,q}x\right\}\right|$$

$$=0.63$$

The right side of inequality is

$$\leq \frac{q(\alpha_2-\alpha_1)}{2[3]_{p,q}}\left[|_{\alpha_1}D_{p,q}u(\alpha_2)|+|^{\alpha_2}D_{p,q}u(\alpha_1)|+\frac{([3]_{p,q}-[2]_{p,q})\big(|_{\alpha_1}D_{p,q}u(\alpha_1)|+|^{\alpha_2}D_{p,q}u(\alpha_2)|\big)}{[2]_{p,q}}\right]$$

$$=0.74$$

So, it's clear

$$0.63<0.74$$

Hence proved.

Example 2: let $u=x^2+2$ is convex function on $[0, 1]\rightarrow\mathbb{R}$ with $p=q=\frac{1}{2}$ *and* $\gamma=2$ then L.H.S of inequality is following

$$\left|\frac{u(\alpha_1)+u(\alpha_2)}{2}-\frac{1}{2(\alpha_2-\alpha_1)}\left\{\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x+\int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2} d_{p,q}x\right\}\right|$$

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$$= \left| \frac{2+3}{2} - \frac{1}{2(1-0)} \left\{ \int_0^{\frac{1}{2}} (x^2+2)_{\alpha_1} d_{p,q}x + \int_{\frac{1}{2}}^{1} (x^2+2)^{\alpha_2} d_{p,q}x \right\} \right|$$

$$= 0.63$$

The R.H.S of inequality is become

$$\leq \frac{q(\alpha_2-\alpha_1)}{2[2]_{p,q}} \left[\left(\frac{[2]_{p,q} |_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma} + ([3]_{p,q} - [2]_{p,q}) |_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma}}{[3]_{p,q}} \right)^{\frac{1}{\gamma}} + \left(\frac{[2]_{p,q} |^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma} + ([3]_{p,q} - [2]_{p,q}) |^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma}}{[3]_{p,q}} \right)^{\frac{1}{\gamma}} \right]$$

$$= 0.81$$

So, it's clear

$$0.63 < 0.81$$

Hence proved.

Example 3: let $u = x^2 + 2$ is convex function on $[0, 1] \to \mathbb{R}$ with p=q=$\frac{1}{2}$ $and\ \gamma = \delta = 2$ then the L.H.S of the inequality is

$$\left| \frac{u(\alpha_1)+u(\alpha_2)}{2} - \frac{1}{2(\alpha_2-\alpha_1)} \left\{ \int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x + \int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2} d_{p,q}x \right\} \right|$$

$$= \left| \frac{2+3}{2} - \frac{1}{2(1-0)} \left\{ \int_0^{\frac{1}{2}} (x^2+2)_{\alpha_1} d_{p,q}x + \int_{\frac{1}{2}}^{1} (x^2+2)^{\alpha_2} d_{p,q}x \right\} \right|$$

$$= 0.63$$

The R.H.S of inequality is become

$$= \frac{q(\alpha_2-\alpha_1)}{2} \left(\frac{1}{[\gamma+1]_{p,q}} \right)^{\frac{1}{\gamma}} \left[\left(\frac{|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\delta} + (p+q-1) |_{\alpha_1}D_{p,q}u(\alpha_1)|^{\delta}}{[2]_{p,q}} \right)^{\frac{1}{\delta}} + \left(\frac{|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\delta} + (p+q-1) |^{\alpha_2}D_{p,q}u(\alpha_2)|}{[2]_{p,q}} \right)^{\frac{1}{\delta}} \right]$$

$Where\, \gamma^{-1} + \delta^{-1} =$

$$= 0.94$$

So, it's clear

$$0.63 < 0.94$$

Hence proved.

Example 4: let $u = x^2 + 2$ is convex function on $[0, 1] \to \mathbb{R}$ with $p = q = \frac{1}{2}$ then the L.H.S of inequality is that

$$\left| u\left(\frac{\alpha_2+\alpha_1}{2} \right) - \frac{1}{2(\alpha_2-\alpha_1)} \left[\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x + \int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2} d_{p,q}x \right] \right|$$

$$= \left| \frac{5}{2} - \frac{1}{2(1-0)} \left\{ \int_0^{\frac{1}{2}} (x^2+2)_{\alpha_1} d_{p,q}x + \int_{\frac{1}{2}}^{1} (x^2+2)^{\alpha_2} d_{p,q}x \right\} \right|$$

$$= 0.27$$

The right side of inequality is

$$\leq \frac{\alpha_2 - \alpha_1}{2} \Bigg[|_{\alpha_1}D_{p,q}u(\alpha_2)| \frac{4p^2q - 3p^2 - 3pq - 3q^2}{4([3]_{p,q}[2]_{p,q})} + |_{\alpha_1}D_{p,q}u(\alpha_1)| \frac{2(2q^3 - 2p^3 + 2p^2q + 2pq^2 + 3p^2}{8([3]_{p,q}[2]_{p,q})} + |^{\alpha_2}D_{p,q}u(\alpha_1)| \frac{4p^2q - 3p^2 - 3pq - 3q^2}{4([3]_{p,q}[2]_{p,q})} + |^{\alpha_2}D_{p,q}u(\alpha_2)| \frac{2(2q^3 - 2p^3 + 2p^2q + 2pq^2 + 3p^2}{8([3]_{p,q}[2]_{p,q})} \Bigg]$$

$$= 0.44$$

So, it's clear

$$0.27 < 0.44$$

Hence proved.

Example 5: let $u = x^2 + 2$ is convex function on [0, 1]→ ℝ with p=q=$\frac{1}{2}$ $and\ \gamma = 2$ then L.H.S of inequality is following

$$\left| u\left(\frac{\alpha_2 + \alpha_1}{2}\right) - \frac{1}{2(\alpha_2 - \alpha_1)} \left[\int_{\alpha_1}^{p\alpha_2 + (1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x + \int_{p\alpha_1 + (1-p)\alpha_2}^{\alpha_2} u(x)^{\alpha_2} d_{p,q}x \right] \right|$$

$$= \left| \frac{5}{2} - \frac{1}{2(1-0)} \left\{ \int_0^{\frac{1}{2}} (x^2+2)_{\alpha_1} d_{p,q}x + \int_{\frac{1}{2}}^{1} (x^2+2)^{\alpha_2} d_{p,q}x \right\} \right|$$

$$= 0.27$$

The R.H.S of inequality is become

$$\leq \frac{\alpha_2 - \alpha_1}{2} \Bigg[\left(\frac{q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}} \times \left(|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma} \frac{q}{8[3]_{p,q}} + |_{\alpha_1}D_{p,q}u(\alpha_1)|^{\gamma} \frac{[3]_{p,q} + q^2}{8([3]_{p,q}[2]_{p,q})} \right)^{\frac{1}{\gamma}}$$

$$+ \left(\frac{2-q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}} \times \left(|_{\alpha_1}D_{p,q}u(\alpha_2)|^{\gamma} \frac{6 - q[2]_{p,q}}{8([3]_{p,q}[2]_{p,q})} + |_{\alpha_1}D_{p,q}u(\alpha_1)|^{\gamma} \frac{5q - 2q^2 - 2}{8[3]_{p,q}} \right)^{\frac{1}{\gamma}}$$

$$+ \left(\frac{q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}} \times \left(|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma} \frac{q}{8[3]_{p,q}} + |^{\alpha_2}D_{p,q}u(\alpha_2)|^{\gamma} \frac{[3]_{p,q} + q^2}{8([4]_{p,q} + q[2]_{p,q})} \right)^{\frac{1}{\gamma}}$$

$$+ \left(\frac{2-q}{4[2]_{p,q}}\right)^{1-\frac{1}{\gamma}} \times \left(|^{\alpha_2}D_{p,q}u(\alpha_1)|^{\gamma} \frac{6 - q[2]_{p,q}}{8([3]_{p,q}[2]_{p,q})} + |^{\alpha_2}D_{p,q}u(\alpha_2)|^{\gamma} \frac{5q - 2q^2 - 2}{8[3]_{p,q}} \right)^{\frac{1}{\gamma}} \Bigg]$$

$$= 0.48$$

So, it's clear

$$0.27 < 0.48$$

Hence proved.

Example 6: let $u = x^2 + 2$ is convex function on [0, 1] with p=q=$\frac{1}{2}$ $and\ \gamma = \delta = 2$ then the L.H.S of the inequality is

$$\left| u\left(\frac{\alpha_2 + \alpha_1}{2}\right) - \frac{1}{2(\alpha_2 - \alpha_1)}\left[\int_{\alpha_1}^{p\alpha_2+(1-p)\alpha_1} u(x)_{\alpha_1} d_{p,q}x + \int_{p\alpha_1+(1-p)\alpha_2}^{\alpha_2} u\,(x)^{\alpha_2} d_{p,q}x\right]\right|$$

$$= \left|\frac{5}{2} - \frac{1}{2(1-0)}\left\{\int_0^{\frac{1}{2}} (x^2+2)_{\alpha_1} d_{p,q}x + \int_{\frac{1}{2}}^{1} (x^2+2)^{\alpha_2} d_{p,q}x\right\}\right|$$

$$= 0.27$$

The R.H.S of inequality is become

$$\leq \frac{\alpha_2 - \alpha_1}{2}\left[q\left(\frac{1}{2^{\gamma+1}[\gamma+1]_q}\right)^{\frac{1}{\gamma}} \times \left(|_{\alpha_1}D_q u(\alpha_2)|\frac{1}{4[2]_{p,q}} + |_{\alpha_1}D_q u(\alpha_1)|\frac{1+2q}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1} |qt-1|^{\gamma} d_q t\right)^{\frac{1}{\gamma}} \times \left(|_{\alpha_1}D_q u(\alpha_2)|\frac{3}{4[2]_{p,q}} + |_{\alpha_1}D_q u(\alpha_1)|\frac{6q-1}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}$$

$$+\left(\frac{1}{2^{\gamma+1}[\gamma+1]_q}\right)^{\frac{1}{\gamma}} \times \left(|^{\alpha_2}D_q u(\alpha_1)|\frac{1}{4[2]_{p,q}} + |^{\alpha_2}D_q u(\alpha_2)|\frac{1+2q}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}$$

$$\left.+\left(\int_{\frac{1}{2}}^{1} |1-qt|^{\gamma} d_q t\right)^{\frac{1}{\gamma}} \times \left(|^{\alpha_2}D_q u(\alpha_1)|\frac{3}{4[2]_{p,q}} + |^{\alpha_2}D_q u(\alpha_2)|\frac{6q-1}{4[2]_{p,q}}\right)^{\frac{1}{\delta}}\right]$$

Where $\gamma^{-1} + \delta^{-1} =$

$$= 0.56$$

So, it's clear

$$0.27 < 0.56$$

Hence proved.

6. CONCLUSIONS

In the (p, q)-calculus framework, we prove new versions of the trapezoidal and midpoint inequalities for differentiable convex functions. Additionally, we use famous Hölder and power mean inequality for (p, q)-differentiable functions to construct (p, q)-midpoint and (p, q)-type trapezoidal inequalities. These novel results have applications in determining certain error boundaries for the trapezoidal and midpoint principles in p, q-integration formulae, which are crucial in numerical analysis. The possibility that post-quantum coordinated convex mappings might lead to new inequality formulations by mathematicians working in this area is an interesting one.

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