On Products of Polynomial Conjugate EP_r Matrices

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Abstract

In this paper we disscuss the product of polynomial conjugate EP_r *(con-*EP_r*) matrices is polynomial con-*EP_r.

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1.Introduction

Throughout this paper we deal with complex polynomial square matrices. An $n\times n$ square matrix $A(\lambda)$ which is a polynomial in the scalar variable λ from a field C represented by $A(\lambda)=A_m\lambda^m+A_{m-1}\lambda^{m-1}+.......+A_1\lambda+A_0$ where the leading coefficient $A_m\neq 0$, A_i's are square matrices in $V_{n\times n}$ is defined a polynomial matrix. Any matrix A is said to be polynomial con-EP_r if $R(A)=R(A^T)$ or equivalently $N(A)=N(A^T)$ or equivalently $AA^+=\overline{A^+A}$ and is said to be polynomial con-EP_r if A is polynomial con-EP_r and $\rho(A)=r$, where $R(A), N(A), \overline{A}, A^T$ and $\rho(A)$ denote the range space, null space, conjugate, transpose and rank of A respectively. $A^\dagger$ denotes the Moore-Penrose inverse of A satisfying the following four equations:(1) $AXA=A$, (2) $XAX=X$, (3) $(AX)^*=AX$, (4) $(XA)^*=XA$ [2]. A^* is the conjugate transpose of A. In general product of two polynomial con-EP_r matrices need not be polynomial con-EP_r. For instance, $\begin{bmatrix}\lambda i & 0\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\ 0 & \lambda i\end{bmatrix}$ are polynomial con-EP_1 matrices, but the product is not polynomial con-EP_1 matrix. The purpose of this paper is to answer the question of when the product of polynomial con-EP_r matrices is polynomial con-EP_r, analogous to that of EP_r matrices studied by [1]. We shall make use of the following results on range space, rank and generalized inverse of a matrix.

(1) $R(A)=R(B)\Leftrightarrow AA^\dagger=BB^\dagger$

(2) $R(A^\dagger)=R(A^*)$

(3) $\rho(A)=\rho(A^\dagger)$

$\Rightarrow\rho(A^T)=\rho(\overline{A})$

(4) $(A^\dagger)^\dagger=A$.

2.ON PRODUCTS OF POLYNOMIAL CONJUGATE EP_r MATRICES

In this section, Explain the product of polynomial conjugate EP_r (con-EP_r) matrices is polynomial con-EP_r

Theorem 2.1

Let A_1 and $A_n (n > 1)$ be polynomial con-EP_r matrices and let $A = A_1A_2A_3......A_n$. Then the following statements are equivalent.

(i) A is polynomial con-EP_r.

(ii) $R(A_1) = R(A_n)$ and $\rho(A) = r$

(iii) $R(A_1^*) = R(A_n^*)$ and $\rho(A) = r$

(iv) $A^\dagger$ polynomial con-EP_r.

Proof:

(i) $\Leftrightarrow$ (ii): Since $R(A) \subseteq R(A_1)$ and $rk(A) = rk(A_1)$ we get $R(A) = R(A_1)$. Similarly $R(A^T) = R(A_n^T)$.

Now, A is polynomial con-EP_r $\Leftrightarrow R(A) = R(A^T)$ and $\rho(A) = r$ (by definition of polynomial con-EP_r)

$\Leftrightarrow R(A_1) = R(A_n^T)$ and $\rho(A) = r$

$\Leftrightarrow R(A_1) = R(A_n)$ and $\rho(A) = r$

(since A_n is polynomial con-EP_r)

(ii) $\Leftrightarrow$ (iii)

$R(A_1) = R(A_n) \Leftrightarrow A_1A_1^\dagger = A_nA_n^\dagger$ (by result (1))

$\Leftrightarrow \overline{A_1A_1^\dagger} = \overline{A_nA_n^\dagger}$

$\Leftrightarrow A_1^\dagger A_1 = A_n^\dagger A_n$ (since A_1, A_n are polynomial con-EP_r)

$\Leftrightarrow R(A_1^\dagger) = R(A_n^\dagger)$ (by results (1)and (4))

$\Leftrightarrow R(A_1^*) = R(A_n^*)$ (by results (2))

Therefore,

$R(A_1) = R(A_n)$ and $\rho(A) = r \Leftrightarrow R(A_1^*) = R(A_n^*)$ and $\rho(A) = r$.

(iv) $\Leftrightarrow$ (i):

$A^\dagger$ is polynomial con-EP_r $\Leftrightarrow R(A^\dagger) = R(A^\dagger)^T$ and $\rho(A^\dagger) = r$ (by definition of polynomial con-EP_r)

$\Leftrightarrow R(A^\dagger) = R(\overline{A})$ and $\rho(A^\dagger) = r$

$\Leftrightarrow R(A^T) = R(A)$ and $\rho(A) = r$

(by results (2)and (3))

$\Leftrightarrow$ A is polynomial con-EP_r.

Hence the theorem.

Corollary 2.2

Let A and B be polynomial con-EP_r matrices. Then AB is a polynomial con-EP_r matrices $\Leftrightarrow$ $\rho(AB) = r$ and $R(A) = R(B)$.

Proof:

Proof follows from Theorem 1 for the product of two matrices A, B.

Remark 2.3

In the above corollary both the conditions that $\rho(AB) = r$ and $R(A) = R(B)$ are essential for a product of two polynomial con-EP_r matrices to be polynomial con-EP_r. This can be seen in the following:

Example 2.4

Let $A = \begin{bmatrix} 1 & \lambda i \\ \lambda i & -1 \end{bmatrix}$, $B = \begin{bmatrix} -\lambda i & 1 \\ 1 & \lambda i \end{bmatrix}$ be polynomial con-EP_r matrices. Here $R(A) = R(B)$, $\rho(AB) \neq 1$ and AB is not polynomial con-EP_1.

Example 2.5

Let $A = \begin{bmatrix} \lambda i & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \lambda i & \lambda i \\ \lambda i & \lambda i \end{bmatrix}$ by polynomial con-EP_r matrices. Here $R(A) \neq R(B)$, $\rho(AB) = 1$ and AB is not polynomial con-EP_1.

Remark 2.6

In particular for $A = B$, corollary 1 reduces to the following.

Corollary 2.7

Let A be polynomial con-EP_r. Then A^k is polynomial con-EP_r $\Leftrightarrow \rho(A^k) = r$.

Theorem 2.8

Let $\rho(AB) = \rho(B) = r_1$ and $\rho(BA) = \rho(A) = r_2$. If AB, B are con$-EP_{r1}$ and A is con$-EP_{r2}$, then BA is con$-EP_{r2}$.

Proof

Since $\rho(BA)=\rho(A)=r_2$,it is enough to show that $N(BA)=N(BA)^T$. $N(A)\subseteq N(BA)$ and $\rho(BA)=\rho(A)$ implies $N(BA)=N(A)$. Similarly $N(AB)=N(B)$. Now,

$$N(BA)=N(A)$$

$=N(A^T)$ (Since A is polynomial con-EP_{r_2})

$\subseteq N(B^TA^T)$

$=N((AB)^T)$

$=N(AB)$ (Since AB is polynomial con-EP_{r_1})

$=N(B)$ (Since $N(AB)=N(B)$)

$=N(B^T)$ (Since B is polynomial con-EP_r)

$\subseteq N(A^TB^T)=N(BA^T)$.

Further, $\rho(BA)=\rho(BA)^T$ implies $N(BA)=N(BA)^T$. Hence the Theorem.

Lemma 2.9

If A, B are polynomial con-EP_r matrices and AB has rank r, then BA has rank r.

Proof:

$$\rho(AB)=\rho(B)-\dim\left(N(A)\cap N(B^*)^\perp\right).$$

Since $\rho(AB)=\rho(B)=r$, $N(A)\cap N(B^*)^\perp=0$

$N(A)\cap N(B^*)^\perp=0 \Rightarrow N(A)\cap N(\overline{B})^\perp=0$ (Since B is polynomial con-EP_r)

$\Rightarrow N(\overline{A})^\perp\cap N(B)=0$

$\Rightarrow N(A^*)^\perp\cap N(B)=0$ (Since A is polynomial con-EP_r).Now,

$$\rho(BA)=\rho(A)-\dim\left(N(B)\cap N(A^*)^\perp\right)=r-0=r$$

Hence the Lemma.

Theorem 2.10

If A, B and AB AB are polynomial con-EP_r matrices, then BA is polynomial con-EP_r.

Proof:

Since A, B are polynomial con-EP_r matrices and $\rho(AB)=r$, by Lemma 1, $\rho(AB)=r$. Now the result follows from Theorem 2, for $r_1=r_2=r$.

Remark 2.11

For any two polynomial con-EP_r matrices A and B, since AB, $\overline{AB}$, $\overline{A^\dagger}B$, $A\overline{B^\dagger}$, $A^\dagger B^\dagger$, $B^\dagger A^\dagger$ all have the same rank, the property of a matrix being polynomial con-EP_r is preserved for its conjugate and Moore-Penrose inverse, by applying Corollary 1 for a pair of polynomial con-EP_r matrices among A, B, $A^\dagger$, $B^\dagger$, $\overline{A}$, $\overline{B}$, $\overline{A^\dagger}$, $\overline{B^\dagger}$ and using the result 2, we can deduce the following.

Corollary 2.12

Let A, B be polynomial con-EP_r matrices. Then the following statements are equivalent.

(i) AB is polynomial con-EP_r matrices.

(ii) $\overline{AB}$ is polynomial con-EP_r matrices.

(iii) $\overline{A^\dagger}B$ is polynomial con-EP_r matrices.

(iv) $A\overline{B^\dagger}$ is polynomial con-EP_r matrices

(v) $A^\dagger B^\dagger$ is polynomial con-EP_r matrices

(vi) $B^\dagger A^\dagger$ is polynomial con-EP_r matrices

Theorem 2.13

If A, B are polynomial con-EP_r matrices. $R(\overline{A})=R(B)$ then $(AB)^\dagger=B^\dagger A^\dagger$.

Proof:

Since A is polynomial con-EP_r and $R(\overline{A})=R(B)$, we have $R(A^\dagger)=R(B)$. That is given $x\in C_n$ (the set of all $n\times 1$ complex matrices) there exists a $y\in C_n$ such that $Bx=A^\dagger y$. Now,

$Bx=A^\dagger y\Rightarrow B^\dagger A^\dagger ABx=B^\dagger A^\dagger AA^\dagger y=B^\dagger A^\dagger y$

$\Rightarrow B^\dagger Bx$

Since $B^\dagger B$ is hermitian, it follows that $B^\dagger A^\dagger AB$ is hermitian. Similarly, $R(A^\dagger)=R(B)$ implies $ABB^\dagger A^\dagger$ is hermitian. Further by result (1), $A^\dagger A=BB^\dagger$. Hence,

$$AB(B^\dagger A^\dagger)AB=ABB^\dagger(BB^\dagger)B$$
$$=AB$$
$$(B^\dagger A^\dagger)AB(B^\dagger A^\dagger)=B^\dagger(BB^\dagger)BB^\dagger A^\dagger$$
$$=B^\dagger A^\dagger$$

Thus $B^\dagger A^\dagger$ satisfies the defining equations of the Moore-Penrose inverse, that is, $(AB)^\dagger=B^\dagger A^\dagger$. Hence the theorem.

Remark 2.14

In the above Theorem, the condition that $R(\overline{A}) = R(B)$ is essential.

Example 2.15

Let $A = \begin{bmatrix} \lambda i & \lambda i \\ \lambda i & \lambda i \end{bmatrix}$ and $B = \begin{bmatrix} \lambda i & 0 \\ 0 & 0 \end{bmatrix}$ Here A and B are polynomial con-EP_1 matrices, $\rho(AB) = 1$, $R(\overline{A}) \neq R(B)$ and $(AB)^{\dagger} \neq B^{\dagger}A^{\dagger}$

Remark 2.16

The converse of Theorem 4, need not be true in general. For,

Let $A = \begin{bmatrix} \lambda i & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & \lambda i \end{bmatrix}$. A and B are polynomial con-EP_r matrices, such that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$, but $R(\overline{A}) \neq R(B)$.

Next to establish the validity of the converse of the Theorem 4, under certain condition, first let us prove a Lemma.

Lemma 2.17

Let $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ be an $n \times n$ polynomial con-EP_r matrix where E is an $r \times r$ matrix and if $[E\, F]$ has rank r then E is nonsingular. Moreover there is an $(n-r) \times r$ matrix K such that $A = \begin{bmatrix} E & EK^T \\ KE & KEK^T \end{bmatrix}$.

Proof:

Since A is polynomial con-EP_r, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is polynomial con-EP_r and $[E\, F]$ has rank r, the product $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} E & F \\ 0 & 0 \end{bmatrix}$ is a product of polynomial con-EP_r matrices which has rank r. Therefore by Lemma 1 the product $\begin{bmatrix} E & F \\ G & H \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix}$ has rank r. Hence there is an $(n-r) \times r$ matrix K and an $r \times (n-r)$ matrix L such that $G = KE$, $F = EL$, and E is nonsingular.

Therefore, $A = \begin{bmatrix} E & EL \\ KE & KEL \end{bmatrix}$

Now, set $C = \begin{bmatrix} I_r & 0 \\ -K & I_{n-r} \end{bmatrix}$ and consider

$$CAC^T = \begin{bmatrix} I_r & 0 \\ -K & I_{n-r} \end{bmatrix}\begin{bmatrix} E & EL \\ KE & KEL \end{bmatrix}\begin{bmatrix} I_r & -K^T \\ 0 & I_{n-r} \end{bmatrix}$$

$$= \begin{bmatrix} E & EL \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & -K^T \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} E & -EK^T + EL \\ 0 & 0 \end{bmatrix}$$

CAC^T is polynomial con-EP_r From $N(A) = N(CAC^T)$ it follows that $EL - EK^T = 0$, and so $L = K^T$, completing the proof.

Theorem 2.18

If A, B are polynomial con-EP_r matrices, $\rho(AB) = r$ and $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$, then $R(\overline{A}) = R(B)$.

Proof:

Since A is polynomial con-EP_r, by Theorem 3 in [3], there is a unitary matrix U such that, $U^TAU = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, where D is $r \times r$ nonsingular matrix.

Set $U^*B\overline{U} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$

$$U^T AB\overline{U} = U^TAUU^*B\overline{U} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$= \begin{bmatrix} DB_1 & DB_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} D & 0 \\ 0 & I_{n-r} \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}$$ has rank r and thus.

$$U^*BAU = U^*B\overline{U}U^TAU = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_1D & 0 \\ B_3D & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I_{n-r} \end{bmatrix}$$ has rank r. It follows that $\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix}$ have rank r, so that B_1 is nonsingular.

By Lemma 2, $U^* B\overline{U} = \begin{bmatrix} B_1 & B_1 K^T \\ KB_1 & KB_1 K^T \end{bmatrix}$, with $\rho(U^* B\overline{U}) = \rho(B_1) = r$. By using Penrose representation for the generalized inverse [4], we get

$$(U^* B\overline{U})^\dagger = \begin{bmatrix} B_1^* P B_1^* & B_1^* P B_1^* K^* \\ \overline{K} B_1^* P B_1^* & \overline{K} B_1^* P B_1^* K^* \end{bmatrix}$$ where

$$P = (B_1 B_1^* + B_1 K^T \overline{K} B_1^*)^{-1} B_1 (B_1^* B_1 + B_1^* K^* K B_1)^{-1}$$

$$U^T B^\dagger U = (U^* B\overline{U})^\dagger = \begin{bmatrix} Q & QK^* \\ \overline{K}Q & \overline{K}QK^* \end{bmatrix}$$ where

$$Q = (I + K^T \overline{K})^{-1} B_1^{-1} (I + K^* K)^{-1}$$

$$U^* A^\dagger \overline{U} = (U^T A U)^\dagger = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$U^T A B\overline{U} = U^T A B\overline{U} (U^T A B\overline{U})^\dagger U^T A B\overline{U}$$

$$= U^T A B\overline{U} (U^T (AB)^\dagger \overline{U}) U^T A B\overline{U}$$ (since U is unitary)

$$= U^T A B\overline{U} (U^T B^\dagger A^\dagger \overline{U}) U^T A B\overline{U}$$

(byhypothesis)

$$= U^T A B\overline{U} (U^T B^\dagger U)(U^* A^\dagger \overline{U}) U^T A B\overline{U}$$

(since U is unitary). On simplification, we get,

$$DB_1 Q B_1 + DB_2 \overline{K} Q B_1 = DB_1$$

$$\Rightarrow DB_1 (I + B_1^{-1} B_2 \overline{K}) Q B_1 = DB_1$$

Since $B_2 = B_1 K^T, QB_1 = (I + K^T \overline{K})^{-1}$. Hence $(I + K^T \overline{K}) = (QB_1)^{-1} = I$. Thus $K^T \overline{K} = 0$ which implies $K^* K = 0$ so that $K = 0$.

$$U^* B\overline{U} = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U^T A U = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow U^* \overline{A}\,\overline{U} = \begin{bmatrix} \overline{D} & 0 \\ 0 & 0 \end{bmatrix}$$

Since $\overline{D}$ and B_1 are $r \times r$ nonsingular matrices we have

$$R(\overline{D}) = R(B_1) \Rightarrow R\left(\begin{bmatrix} \overline{D} & 0 \\ 0 & 0 \end{bmatrix}\right) = R\left(\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\Rightarrow R(U^* \overline{A}\,\overline{U}) = R(U^* B\overline{U})$$

$$\Rightarrow R(\overline{A}) = R(B).$$

Hence the Theorem.

Theorem 2.19

Let A, B are polynomial con-EP_r matrices, $\rho(AB) = r$ and $(AB)^\dagger = B^\dagger A^\dagger$, then AB is polynomial con-EP_r.

Proof:

$R(B) = R(B^T)$ (Since B is polynomial con-EP_r)

$\Rightarrow R(\overline{B}) = R(B^*)$

$= R(B^* A^*)$ (Since $= R(B^* A^*) \subseteq R(B^*)$)

and $\rho(AB)^* = \rho(AB) = r = \rho(B^*)$

$= R(AB)^* = R(AB)^\dagger$ (by result (2))

$= R(A^\dagger B^\dagger)$ (by hypothesis)

$\subseteq R(A^\dagger) = R(A^*) = R(\overline{A})$ (by result (2)

and A is polynomial con-EP_r).

$\Rightarrow R(\overline{B}) = R(\overline{A}) \Rightarrow R(B) = R(A)$

Since $\rho(AB) = r$ $R(B) = R(A)$, by Corollary 1, AB is polynomial con-EP_r.

Hence the Theorem.

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