Optimal Spline Based Gas-Lift Allocation Using Lagrange's Multiplier

Peter Amoako-Yirenkyi [1], Godfrey E. Ntherful[2], Gabriel O. Fosu[3], Gideon K. Gogovi[4], Justice K. Appati[5]

Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana[1-5]

Abstract

An accurate prediction of the production rate of fluids from a reservoir into the wellbore is essential for efficient artificial lift application in an oil field. We find an accurate model to determine the optimal distribution of the available lift gas among a group of five oil wells on continuous gas lift in order to maximize the total oil production. The optimization problems were then solved using LaGrange multiplier method subject to the constraint of limited gas supply. The rational function and a modified cubic spline interpolation was used to fit the gas injection data. The total optimum oil production rate for data fitting with the spline based function is found to be higher than the total optimum oil production rate of the rational function. The optimal value of the spline based function was found to be twice that of the rational function.

Keywords: Cubic Spline, Gas lift, Oil Wells, Rational Function

1 Introduction

Most oil producing wells flow naturally without artificial stimulation when they are first drilled. The flow rate depends on the energy level of the reservoir and the energy losses of the fluid as they flow from the reservoir towards the surface facilities. As the wells mature the energy level falls and this affects the production rate to fall to a level that is economically no longer profitable. In order to increase production flow rate, artificial processes applied to either increase the production level or to facilitate the flow of the hydrocarbons. These artificial lift supplements the natural drive effects on pressure maintenance and displacement by employing water injection/water flooding or natural gas injection and rod pumps.

Gas lift technique drive mechanism is identified by the presence of a gas cap over an oil zone in an oil reservoir. The energy to produce the fluid comes from the expansion of the gas in the gas cap and the expansion of the gas liberated from the liquid as the pressure declines. Gas lift can be controlled for a single well to optimize production, and to reduce slugging effects where the gas droplets collect to form large bubbles that can upset production. Injected gas aerates the fluid to reduce its density; the formation pressure is then able to lift the oil column and forces the fluid out of the well-bore [1].

Gas lift optimization can be traced to a study by Redden et al.[2], they calculated optimum distribution of available lift gas to a group of gas lifted wells based on each well's contribution to the profit of the system. Kanu [3], proposed gas lift production optimization with data gathering, systems analysis, gas allocation, gas lift valve placement and evaluation and implementation. Well performance was reviewed by plotting tubing and casing pressures before implementing allocation calculations.

Systems analysis techniques were applied by Amondin and Jackson[4] for optimizing gas lift allocation in a group of gas lifted wells. An optimization software was used to determine the optimal gas allocation rates from the analysis of well performance curves modeled with a polynomial function. The optimization algorithm also handled the choke settings in surrounding naturally flowing wells. Palke and Home [5]

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applied nonlinear optimization algorithms to a field model that was composed of a reservoir model, a well model with gas lift, a choke model and a separator model. The combination of the production parameters such as tubing diameter, separator pressure, gas injection depth and volume of gas injection was looked for optimizing the net present value of the model. The genetic algorithm optimization techniques were found to be both stable and efficient to address these sorts of optimization problems. Laing [6] described production optimization from gas lift wells by conducting special training for operators, analyzing flowing pressure and temperature surveys, replacing defective gas lift valves, measuring correctly injection gas flow rate to each well, twinning surface flow line and improving gas lift design techniques.

Allah [7] discussed the use of a Multi-phase Flow Meter to optimize gas lift field operations. This in particular compares analysis methods individual well's performance using multi flow meter versus Standard Nodal Analysis. Hatton and Potter [8] used SAS/OR optimization techniques to provide quick results using a scalable solution. A description of a non-linear optimization problem with constraints associated with the optimal distribution of the lift gas was given by [9]. They developed a non-linear objective function using a simple dynamic model of the oil field where the decision variables represent the lift gas flow rate set points of each oil well of the field.

Though many mathematical methods have been developed by various researchers, there has not been any study on the C J C Kruger cubic spline interpolation method. In this study we determine an accurate model for exact prediction of oil using the the modified cubic spline method. We compare this method to the rational function method. We also find a suitable approximation method that will best lead the Lagrange to find the appropriate optimum value.

2 Methodology

The methodology is based on rational function and cubic spline interpolations for curve fitting and Lagrange's multiplier for the optimization. We used the LaGrange's multipliers to find equal point for gas allocation considering limited gas available.

2.1 Curve Fitting of Data using Rational Function Method

The gas injection and oil production rates are fit by using Rational function for each well. The oil production rate (σ) measured in STB/D is considered as dependent variable where the gas injection rate (α) measured in MSCF/D is the independent variable. Coefficients of the function a, b and c are determined by the least square method. The rational function is defined as $\sigma = (a + c\alpha)/(1 + b\alpha)$.
The function to be minimized for the determination the least square coefficients, is defined as follows [10, 11]:

$$\begin{aligned} SMIN &= \sum_{i=1}^{5}\left(\sigma_i - \frac{(a + c\alpha_i)}{(1 + b\alpha_i)}\right)^2 \\ &= \sum_{i=1}^{5}\frac{(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)^2}{(1 + b\alpha_i)} \\ &= \sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)^2 \end{aligned} \tag{1}$$

$1 + b\alpha_i$ is initially set to unity. The necessary conditions for minimizing $SMIN$ are given as:

$$\begin{aligned} \frac{\partial SMIN}{\partial a} &= 2\sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)(-1) = 0 \\ \frac{\partial SMIN}{\partial b} &= 2\sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)(\alpha_1\sigma_1) = 0 \\ \frac{\partial SMIN}{\partial c} &= 2\sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)(-\alpha_1) = 0 \end{aligned} \tag{2}$$

The results for the least square coefficients are:

$$a = \frac{D_a}{D};\ b = \frac{D_b}{D};\ c = \frac{D_c}{D}$$

where

$$D_a = \begin{vmatrix} \sum_{i=1}^{5}\sigma_i & -\sum_{i=1}^{5}\alpha_i\sigma_i & \sum_{i=1}^{5}\alpha_i \\ \sum_{i=1}^{5}\alpha_i\sigma_i & -\sum_{i=1}^{5}\alpha_i^2\sigma_i & \sum_{i=1}^{5}\alpha_i^2 \\ \sum_{i=1}^{5}\alpha_i\sigma_i^2 & -\sum_{i=1}^{5}\alpha_i^2\sigma_i^2 & \sum_{i=1}^{5}\alpha_i^2\sigma_i \end{vmatrix}$$

$$D_b = \begin{vmatrix} 5 & -\sum_{i=1}^{5}\sigma_i & \sum_{i=1}^{5}\alpha_i \\ \sum_{i=1}^{5}\alpha_i & -\sum_{i=1}^{5}\alpha_i\sigma_i & \sum_{i=1}^{5}\alpha_i^2 \\ \sum_{i=1}^{5}\alpha_i\sigma_i & -\sum_{i=1}^{5}\alpha_i\sigma_i^2 & \sum_{i=1}^{5}\alpha_i^2\sigma_i \end{vmatrix}$$

$$D_c = \begin{vmatrix} 5 & -\sum_{i=1}^{5}\alpha_i\sigma_i & \sum_{i=1}^{5}\sigma_i \\ \sum_{i=1}^{5}\alpha_i & -\sum_{i=1}^{5}\alpha_i^2\sigma_i & \sum_{i=1}^{5}\alpha_i\sigma_i \\ \sum_{i=1}^{5}\alpha_i\sigma_i & -\sum_{i=1}^{5}\alpha_i^2\sigma_i^2 & \sum_{i=1}^{5}\alpha_i\sigma_i^2 \end{vmatrix}$$

The sum of the squares of the errors between the data and the function is calculated for wells is given as:

$$S = \sum_{i=1}^{5}(\sigma_i - \frac{a + c\alpha_i}{1 + b\alpha_i})^2$$

The sum of the squares of the errors between the data and the mean oil output rate is calculated for the wells is also given as:

$$S_B = \sum_{i=1}^{5}(\sigma_i - \frac{\sum_{i=1}^{5}\sigma_i}{5})^2$$

2.2 Cubic Spline Interpolation

The cubic spline is considered as an alternative to the least square rational function for fitting well data. Due to its stable and smooth characteristics, it fits the data very well and represents true well behaviour. So the cubic spline is an appropriate choice to model the well gas injection and oil output data. Kruger's [12] proposed Constrained Cubic Splines is applied in this study.

The principle behind the proposed constrained cubic spline is to prevent overshooting by sacrificing smoothness. This is achieved by eliminating the requirement for equal second order derivatives at every point and replacing it with specified first order derivatives. Thus, Kruger[12] proposed Constrained Cubic Spline is as follows:

$$f'(\alpha_i) = \frac{2}{\frac{\alpha_{i+1}-\alpha_i}{\sigma_{i+1}-\sigma_i} + \frac{\alpha_i-\alpha_{i-1}}{\sigma_i-\sigma_{i-1}}}$$

$$f_i''(\alpha_{i-1}) = -\frac{2[f_i'(\alpha_i) + 2f_i'(\alpha_{i-1})]}{(\alpha_i - \alpha_{i-1})} + \frac{6(\sigma_i - \sigma_{i-1})}{(\alpha_i - \alpha_{i-1})^2}$$

Hence, a third degree polynomial constructed between each point is generally defined as:

$$f_i(\alpha) = a_i + b_i\alpha + c_i\alpha^2 + d_i\alpha^3$$

Where the actual parameters (a_i, b_i, c_i and d_i) for each of the cubic spline equations are found directly without solving a system of equations and this permits analytical integration of the data.

2.3 Lagrange Multiplier with Rational Function

To find the optimum point of gas injection rate and oil output rate for each well after fitting the gas in and oil out data, the Lagrange optimization is carried out. The optimum oil rate in each well is expressed as a rational function of its optimum gas injection rate as [10, 11]

$$\sigma_i = \frac{a_i + c_i\alpha_i}{1 + b_i\alpha_i}$$

where i = 1,$\cdots$,5 is the i^{th} well, σ_i are the optimum oil output rates and α_i are the optimum gas injection rate for wells 1,$\cdots$, 5. Whiles the respective least square coefficients a_i, b_i and c_i are determined by Matlab simulation. The optimization is subject to a linear equality constraint regarding the availability of limited gas for injection. Let the total amount of gas available be N measured in MSCF/D. Hence the constraint equation is $\alpha_i = N$ where n is the number of wells, with the constraint function defined as $\emptyset = \alpha_i - N = 0$. The LaGrange Multiplier relating the partial derivative of the oil rate in each well to the partial derivative of the constraint function is:

$$\frac{\partial\sigma_i}{\partial\alpha_i} = \lambda\frac{\partial\emptyset}{\partial\alpha_i}$$

where i represents the well number and (λ) is the LaGrange Multiplier. The partial derivatives these equations yields

$$\lambda b_i^2\alpha_i^2 + 2\lambda b_i\alpha_i + \lambda - c_i + a_ib_i = 0$$

where α_i is the expression of gas injection rates of the i^{th}, well. This set of non-linear equation is solved using Newton's method of solving algebraic set of equations. The total optimum gas injection rate is $\sum_{i=1}^{n}\alpha_i$ which is equal to the total available gas for injection. Using the least square coefficients and the optimum gas injection rate for the wells, the optimum oil output rate of the wells are calculated from the following equation:

$$\sigma_i = \frac{a_i + c_i\alpha_i}{1 + b_i\alpha_i}.$$

Hence the total optimum oil output rates of the wells is calculated from $\sum_{i=1}^{n}\sigma_i$.

2.4 Lagrange Multiplier with Cubic Spline Function

After fitting the gas injection and oil output rates of the wells with Cubic Spline functions, the optimum rates of gas injection and oil output in each of the wells are determined using the LaGrange Multiplier method. The optimum oil rate in each well is expressed as a cubic spline function of its optimum gas injection. Cubic Spline function for the five wells is given as:

$$\sigma_j = a_i + b_i\alpha_j + c_i\alpha_j^2 + d_i\alpha_j^3 \quad \forall j = 1, \cdots, 5$$

The analysis is subject to a constraint regarding the availability of limited gas for injection. The total amount of gas available for injection is N MSCF/D. So the constraint equation is defined as follows: $\alpha_i = N$ and a constraint function is defined as $\emptyset = \alpha_i - N = 0$. The LaGrange Multiplier relating the partial derivative of the oil rate in each well to the partial derivative of the constraint function are defined as:

$$\frac{\partial\sigma_i}{\partial\alpha_i} = \lambda\frac{\partial\emptyset}{\partial\alpha_i}$$

resulting to the non-linear equation $3d_i\alpha_j^2 + 2c_i\alpha_j + b_i - \lambda = 0$. This leads us to an algebraic set of n nonlinear equations in 5 unknowns are formed which can be written in the general form as $f_i(\alpha_1, \alpha_2, ..., \alpha_5) = 0$ for $i = 1, 2, ...5$. To calculate Lagrange's multiplier and the optimum gas injection rates of the wells, the Broyden's iterative method for solving algebraic set of nonlinear equations was employed for the analysis[10, 11].

3 Analysis and Results

The initial guess used is the real well data for gas injection rates in wells one to five. These are shown in Table 6 in the appendix. The numerical simulations were performed using Matlab. The production optimization of continuous flow gas lifted wells is carried out using the LaGrange Multiplier method subject to limited gas supply.

3.1 Curve fitting

The data points and the oil output rates calculated from the rational functions are shown in Tables 1 and 5. The oil production rates from data and calculated oil output rate from the rational function fits and cubic spline interpolation model are plotted against gas injection rates in figures 1, **??** and 2. These result illustrates that the cubic spline interpolation model is better as compared to the rational function for fitting the well data since they are flexible to handle and do represent the true well behaviour and fits the data very well.
In order to check the validity of the mathematical model, the continuous gas lift well in table 6 was compared with the results obtained by the interpolation methods in tables 1 and 5. It was noticed that the cubic spline gives a better results. Hence, the productions will take place according to the cubic spline function since it represents real data of the wells.

3.2 Optimization Results

Well data are generated by using a well performance software program as stated by Jamal [13] and fit with cubic spline and rational functions by the least square method. The optimum operating conditions are based on the concept that for each well incremental oil production due to a unit increase in gas injection should be equal. So the optimum operating points are nothing but equal slope points.
The optimum gas injection and oil production rates of all wells for rational function data fit and the value of the LaGrange Multiplier are illustrated in Table 2. The optimum gas injection and oil production rates of all wells for cubic spline function data fit and the value of the LaGrange Multiplier are tabulated in Table 3.

Table 1: Oil output rates from data and the two interpolation methods for well one.

α, Gas Injection (MSCF/D)	σ, Oil output (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
1.75	67	69.62998011	67
115	195	186.5579756	195
385	257	263.3390997	257
735	286	292.660964	286
1290	317	309.8119404	317

Table 2: Optimum gas injection and oil output rates for rational function data fit.

λ	0.202070231			
Well	Optimum gas injection rates (MSCF/D)		Optimum oil output rates (STB/D)	
One	α_1	295.091	σ_1	248.2594091
Two	α_2	125.9270512	σ_2	152.9316134
Three	α_3	119.9383631	σ_3	139.2507759
Four	α_4	177.7020761	σ_4	207.8055049
Five	α_5	281.34129	σ_5	291.5548112
	Total	1000	Total	1039.802114

Table 3: Optimum gas injection and oil output rates for cubic spline function data fit.

Lambda λ	-5.6569			
Well	Optimum gas injection rates (MSCF/D)		Optimum oil output rates (STB/D)	
One	α_1	296.3784	σ_1	502.1615
Two	α_2	382.2095	σ_2	132.1152
Three	α_3	-192.4427	σ_3	386.3273
Four	α_4	272.3827	σ_4	516.4365
Five	α_5	241.4726	σ_5	560.4919
	Total	1000	Total	2097.5324

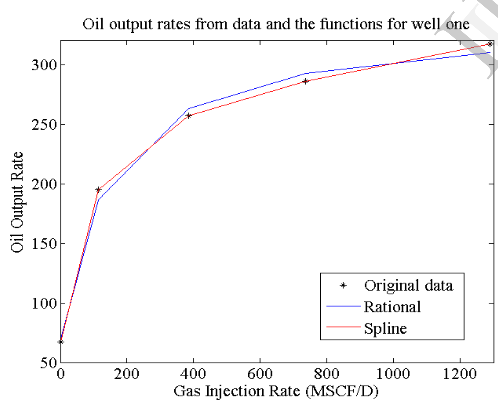

Figure 1: Plot of Oil Output against Gas Injection for Well One

Examining the figures, the overall performance of the cubic spline is most appropriate. As close as all of the curves do follow the data quite well. The rational function is less predictable although it seems to have a good fit to some of the production data. Hence, from visual inspection the cubic spline function had the best match.

Though the rational function gave results closer to real data of the well, these productions do not represent optimum operating conditions for the rational model. Rather, the productions will take place according to the cubic spline since it represents real data of the wells.

Curve fitting of well data with the cubic spline function gives accurate predictions since it gives the same results as the well data. The other model, the rational function though does not give poor fit, its results have some variations with the real well data. So productions do not represent optimum operating conditions for the rational model. This creates some differences between the cubic spline optimum oil outputs and the actual oil outputs at the rational function optimum gas injection rates. These differences which are defined as lost productions, are presented in Table 4.

Table 4: Lost productions due to wrong predictions.

Well	Rational optimum gas rates (MSCF/D)	Actual outputs (STB/D)	Cubic Spline optimum oil rates (STB/D)	Lost production (STB/D)
One	295.091	248.2594091	502.1615	253.9021
Two	125.9270512	152.9316134	132.1152	-20.8164
Three	119.9383631	139.2507759	386.3273	247.0765
Four	177.7020761	207.8055049	516.4365	308.6310
Five	281.34129	291.5548112	560.4919	268.9371
Total	1000	1039.802114	2097.5324	1057.7303

4 Conclusion

In this study we determined an accurate model for finding an optimum amount of gas to maximize oil production and carry out an optimization to cater for the oil loses in order to enhance production. Well data were fitted with cubic spline interpolation method and rational function for least squares method and optimum gas injection and oil output rates were determined for both functions. Total optimum oil production rate for data fitting with cubic spline function is found to be 101.72% higher than the total optimum oil production rate for data fitting with rational function. The cubic spline algorithm was the most suitable for adaptation with good potential and has therefore proven to be a fast algorithm suitable for the purpose of this study.

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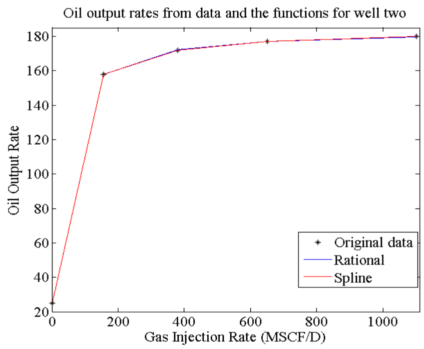
Table 5: Oil output rates from data and the two interpolation methods.

α, Gas Injection rate (MSCF/D)	σ, Oil output rate (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
Well two			
2	25	25.00278081	25
155	158	157.8739039	158
380	172	172.3137542	172
650	177	176.993773	177
1100	180	179.8157803	180
Well three			
1.75	27	26.99631064	27
95	133	133.2076266	133
290	156	155.3633103	156
575	162	162.0617496	162
1050	165	165.3710036	165
Well Four			
1.8	49	48.99256641	49
20	192	192.1627265	192
350	228	227.5843113	228
780	241	241.0795404	241
1350	246	246.1808397	246
Well Five			
1	82	26.99631064	82
95	216	133.2076266	216
320	303	155.3633103	303
750	335	162.0617496	335
1380	346	165.3710036	346

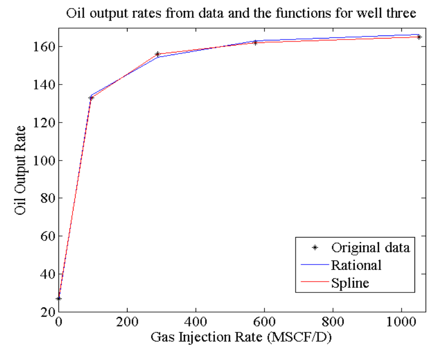
Table 6: Gas injection and Oil output rate [13]

Well one		Well two		Well three		Well four		Well five	
α,	σ,	α,	σ,	α,	σ,	α,	σ,	α,	σ,
1.75	67	2	25	1.75	27	1.8	49	1	82
115	195	155	158	95	133	120	192	95	216
385	257	380	172	290	156	350	228	320	303
735	286	650	177	575	162	780	241	750	335
1290	317	1100	180	1050	165	1350	246	1380	346

α, is Gas injection rate (MSCF/D) and σ, is the Oil output rate (STB/D)



Well two



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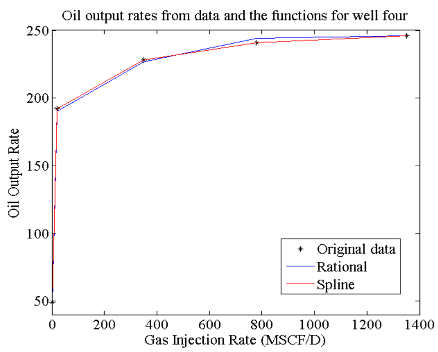
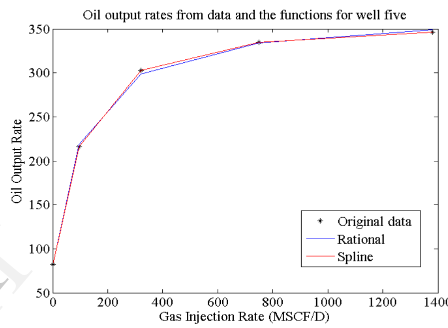


Well four



Well five

Figure 2: Plot of Oil Output versus Gas Injection