Properties of Bondage Arcs

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Abstract:- In this paper, we de ne the non-bondage number b_n(G) for any fuzzy graph and de ne fuzzy strong line graph. A characterization is obtained for fuzzy strong line graphs L_s(G) such that L_s(G) is tree . A necessary condition for a fuzzy double strong line graph of cycle is a fuzzy trees and the exact value of b_n(G) for any graph G is found. Moreover we de ne neighbourhood extension also analysis it extendable by using bondage arcs and relationships between b_n(G) and b.

Keywords: (G)Minimum dominating set , maximum non-bondage number b_n(G) , minimum bondage number b(G); L_s(G) is strong line graph,L_s(G) is double strong line grpah .G_E neigh-bour extendable

1 Introduction

Fuzzy graph theory was introduced by A. Rosenfeld [8] in 1975. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of neural networks, expert systems, information theory, cluster analysis, medical diagnosis, control theory, etc. Sunil Mathew, Sunitha M.S [10] has obtained the fuzzy graph-theoretic concepts like f- bonds, paths, cycles, trees and connectedness and established some of their properties. V.R. Kulli and B. Janakiram [7] have established the non-bondage number of a graph. First we give the definitions of basic concepts of fuzzy graphs and define the non-bondage and it is properties. All graphs consider here are finite, undirected, distinct labeling with no loop or multi arc and p nodes and q (fuzzy) arcs. Any undefined term in this paper may be found in Harary[5]. Among the various applications of the theory of domination that have been considered, the one that is perhaps most often discussed concerns a communication network. Such a network consists of existing communication links between a fixed set of sites. The problem is to select a smallest set of sites at which to place transmitters so that every site in the network that does not have a transmitter is joined by a direct communication link to one that does have a transmitter. This problem reduces to that of finding a minimum dominating set in the graph corresponding to the network. This graph has a node representing each site and an arc between two nodes iff the corresponding sites have a direct communications link joining them. To minimize the direct communication links in the network, we introduce the following section.

2 Preliminaries

A fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0, 1]$. A fuzzy relation on V is a fuzzy subset of $E(V \times V)$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of function $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where $\mu(u, v) \leq \sigma(u)\sigma(v)$for all u, v ϵV. The underlying crisp graph of G=(σ, μ) is denoted by $G^* = (V, E)$, where $V = \{u\epsilon V : \sigma(u) > 0\}$ and $E = \{(u, v)\epsilon V \times v : \mu(u, v) > 0\}$. The order $P = \sum_{v\epsilon D} \sigma(v)$. The graph $G = (\sigma, \mu)$ isdenoted by G, if unless otherwise mentioned. Let be a fuzzy graph on. The degree of a vertex u is $d_G(u) = \sum\limits_{(u \neq v)} \mu(uv)$. The minimum degree of G is $\delta(G) = \wedge\{d_(G)(u), v\epsilon V\}$and the maximum degree of G is $\delta(G) = \vee d_(G)(u), \forall\epsilon V$ The strength of connectedness between two nodes u and v in a fuzzy graph G is define as the maximum of the strength of all paths between u and v and is denoted by $CONN_G(u, v)$. A u-v path P is called a strongest path if its strength equals $CONN_G(u, v)$. The fuzzy graph $H = (\tau, \rho)$ is called a fuzzy sub graph of G if $\tau(x) \leq \sigma(x)$for all $x\epsilon V$and $\rho(x, y) \leq (x, y)$ for all$(x, y)\epsilon V$. A fuzzy sub graph H =(τ, ρ) is said to be a spanning fuzzy sub graph of G, if $\tau(x) = \sigma(x)$for all x. A fuzzy G is said to be connected if there exists a strongest path A path P of length n is a sequence of distinct nodes u_0u_1, u_2, u_n such that $(u_{i-1}, u_i) > 0$ and degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and it is a fuzzy cycle if there is more than one weak arc. Let u be a node in fuzzy graphs G then $N(u) = \{v : (u, v)\}$ is strong arc}is called neighborhood of u and N[u] = N(u)$\bigcup u$ is called closed neighborhood of u. Neighborhood degree of the node is defined by the sum of the weights of the strong neighbor node of u is denoted by $d_s(u) = \sum\limits_{v\epsilon N(u)} \sigma(v)$

3 Fuzzy dominating set

Definition 3.1. Let G be a fuzzy graph and u be a node in G then there exist a node v such that (u, v) is a strong arc then u dominates v .

Definition 3.2. Let G be a fuzzy graph. A subset D of V is said to be a fuzzy dominating set if for every node $v\epsilon V \setminus D$, there exists $u\epsilon D$ such that u dominates v.

Definition 3.3. The domination number of G is the minimum cardinality taken over all dominating sets in G and is denoted by $\gamma(G)$,where $\gamma(G) = \sum_{v\epsilon D} \sigma(v)$. A dominating set with cardinality $\gamma(G)$ is calledγ - set of G.

4 Fuzzy non bondage number

Definition 4.1. The bondage number b(G) of a fuzzy graph G(V, E, σ, μ)) is minimum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that
$CONN_{G-(x,y)}(u, v) < CONN_G(u, v)$ for all $u\epsilon V - \gamma(G)$ and a v $\epsilon\gamma(G)$. Here $\gamma(G)$ represent minimum dominate set

Definition 4.2. The non-bondage number $b_n(G)$ of a fuzzy graph G(V, E, σ, μ)) is maximum number of fuzzy arcs among all sets of arcs X = (x,y) sub set of E such that
$CONN_{G-(x,y)}(u, v) = CONN_G(u, v)$ for all $u\epsilon V - \gamma(G)$ and a v $\epsilon\gamma(G)$.Here $\gamma(G)$ represent minimum dominate set

Theorem 4.3. *For any fuzzy graph G,*

$$b_n(G) = q - p + \gamma(G) \tag{1}$$

, where q is total number of fuzzy arcs and p is total number of node.

Theorem 4.4. *For any graph G.*

$$b_n(G) \leq q - \Delta_n \tag{2}$$

where Δ_n is number total number of strong arcs in Δ of G .

Theorem 4.5. *For any fuzzy graph, (x, y) is a non bondage iff (x, y) is a weakest arc of any cycle.*

Remark 1. Let G: (σ, μ)be a fuzzy graph such that $G^* : (\sigma^*, \mu^*)$ is a cycle and Let t = $min\{\mu(u, v) : \mu(u, v) > 0\}$.Then all arcs (u,v) of G such that$\mu(u, v) > t$ are fuzzy bondage of G.
$\mu(u, v) = t$ is only non-bondage of G.

Theorem 4.6. *Let G:(σ, μ) be fuzzy graph such that $G^* : (\sigma^*, \mu^*)$ is a cycle. Then a node is a not fuzzy cut node of G iff it is incident with a non-bondage arc.*

Theorem 4.7. *If G is a fuzzy forest, the arcs of F are not fuzzy non -bondage of G.*

Theorem 4.8. *A complete fuzzy graph with n notes has n-1 non bondage.*

Theorem 4.9. *A complete fuzzy graph has no fuzzy cut nodes.*

5 Exact values of b_n (G) for some standard graphs

Proposition 5.1. *If $P_p$is a path with $p \geq 4$ nodes, then $b_n(P_p) = \lceil p/3 \rceil - 1$.*

Proposition 5.2. *If C_p is a cycle with $p \geq 3$ nodes, then $b_n(C_p) = \lceil p/3 \rceil$.*

Proposition 5.3. *If K_p is a complete graph p$\geq$ 3 nodes, then $b_n(K_p) = \frac{(p-1)(p-2)}{2}$.*

Proposition 5.4. *If $K_{m,n}$ is a complete bipartite graph ,then $b_n(K_{m,n}) = mn - m - n + 2$*

Proposition 5.5. *If W_p is a wheel with $p \geq 4$ nodes, then $b_n(W_p) = p - 1$.*

Proposition 5.6. *For any tree T, $b_n(T) = \gamma(T) - 1$.*

6 Relationships between b_n(G) and b

Theorem 6.1. *Let T$\neq$ P_4 be a tree with at least two cut nodes .then*

$$b_n(G) \geq b(T) \tag{3}$$

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Theorem 6.2. *For any fuzzy graph*

$$b(G) \leq b_n(G) + 1 \tag{4}$$

Theorem 6.3. *If G be a cycle graph then*

$$b_n(\bar{G}) + b_n(G) \leq \frac{p(p-3)}{2} \tag{5}$$

, $p \geq 4$

Proof. by theorem 6 $b_n(G) = q - p + \gamma(G)$
$b_n(\bar{G}) = \bar{q} - p + \gamma(\bar{G})$
$b_n(\bar{G}) + b_n(G) = q - p + \gamma(G) + \bar{q} - p + \gamma(\bar{G})$
$= q + \bar{q} - 2p + (\bar{G}) + \gamma(G)$
$= \frac{(p(p-1))}{2} - 2p + \gamma(\bar{G}) + \gamma(G)$
$\leq \frac{(p(p-1))}{2} - 2p + p$
$\leq \frac{(p(p-3))}{2}$. □

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Theorem 6.4. *For any graph G,*

$$b_n(\bar{G}) + b_n(G) \leq \frac{((p-1)(p-2))}{2} \tag{6}$$

Proof. by Theorem 7 $b_n \leq q - \Delta_n$,then
$b_n(\bar{G}) + b_n(G) \leq \bar{q} + q - (\Delta_n + \delta_n)$
$= \frac{(p(p-1))}{2} - (\Delta_n + \delta_n)$
$\leq \frac{(p(p-1))}{2} - (p-1)$
$\leq \frac{((p-2)(p-1))}{2}$. □

Theorem 6.5. *If G be a cycle graph then*

$$b(\bar{G}) + b(G) \leq \frac{(p(p-3))}{2} + 2 \tag{7}$$

Proof. by Theorem 9
$b(G) \leq b_n(G) + 1$
$b(\bar{G}) + b(G) \leq b_n(\bar{G}) + b_n(G) + 2$
by Theorem 10
$b(\bar{G}) + b(G) \leq \frac{(p(p-3))}{2} + 2$ □

Theorem 6.6. *Any graph G,*

$$b(\bar{G}) + b(G) \leq \frac{((p-1)(p-2))}{2} + 2 \tag{8}$$

Theorem 6.7. *If G be a tree then*

$$b_n(\bar{G}) + b_n(G) \geq \gamma(\bar{G}) + \gamma(G) - 2, \tag{9}$$

if $p \geq 4$

Proof. $b_n(G) \geq \gamma(G) - 1$, then
$b_n(\bar{G}) + b_n(G) \geq \gamma(G) + \gamma(G) - 2.$ □

7 Block

Definition 7.1. A connected fuzzy graph is called block if all nodes are satisfies the condition $CONN_{G-v}(u, v) = CONN_G(u, v)$ for every u, v in G..

8 Strong line graph

Definition 8.1. Given a fuzzy graph G, its strong line graph $L_s(G)$ is a fuzzy graph, $L_s(G)$ is a graph G such that
Each node of $L_s(G)$ represents an arc of G; and Two nodes of $L_s(G)$ are adjacent if and only if their corresponding arcs are strong and share a common end point in G.

Definition 8.2. Given a fuzzy graph G, its double strong line graph $L_s^*(G)$ is a fuzzy graph, $L_s^*(G)(\tau, \rho)$ is a graph G such that
Each node of $L_s^*(G)$ represents a strong arc of G; and Two nodes of $L_s^*(G)$ (say u and v) are adjacent if and only if their corresponding arcs are strong and share a common end point in G say x.$\rho(u, v) = \mu(k, x) \wedge \mu(x, w)$

Theorem 8.3. *For any cycle fuzzy graph, then Ls(G) has one isolate node iff G has a non bondage arc.*

Proof. Let $L_s(G)$ has one isolated node, so G has one weakest arc (x, y) by theorem 4.5 then (x, y)is non bondage. Conversely, let G has a non-bondage arc (x, y), clearly (x, y) weakest arc and node x and y does not common node for two strong arcs. So corresponding vertex of arc (x, y) in $L_s(G)$ is isolated. □

Theorem 8.4. *Let $L_s(G)$ be fuzzy black graph such that $L_s(G)$ is a tree. Then a node in $L_s(G)$ iff G has non-bondage or a arc adjacent with a non-bondage arc.*

Proof. Given $L_s(G)$ be fuzzy block then $CONN_{G-v}(u,w) = CONN_G(u,w)$ by definition here node of $L_s(G)$ is a arc of G, so we clearly that $CONN_{G-(x,y)}(r,v) = CONN_G(r,v)$, converse true trivially. □

Theorem 8.5. *A complete fuzzy graph of $L_s(G)$ is not a complete.*

Proof. Given that G is compete fuzzy graph so there exist a cycle in G and every node of even pair does not adjacent with odd pair so corresponding nodes not adjacent in $L_s(G)$. □

Theorem 8.6. *Let G be a path graph with n nodes then $L_s^*(G)$ has n-2 arcs.*

Proof. Given G be a path with n nodes then n-1 arcs so $L_s^*(G)$ has a path with n-1 nodes so it has n-2 arcs. □

Theorem 8.7. *Let G be complete graph with n nodes then $L_s^*(G)$ is 2(n-2) regular graph.*

Proof. Given G be complete graph so every node has n-1 strong arc then every arc adjacent with 2n-4, so $L_s^*(G)$ has 2(n-2) regular graph. □

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9 Neighbourhood Extension

Definition 9.1. Let G be graph and$S_i \subseteq V$, each S_i is collection of each node in G and let $G_E(\tau, \rho)$ be underlying crisp graph. If $G_E(\tau, \rho)$said to be Neighbourhood extension, then satisfied following condition.

- Each node of G_E represents an strong neighbourhood set of G
- Two nodes of G_E are adjacent iff their correspond neighbour set have at least one common node
  where$\rho(S_i, S_j) = min\{\mu(x, v_i), \mu(v_j, x)/x\epsilon S_i \cap S_j\}$

Definition 9.2. Let G* be Connected graph and if $G_E \cong G^*$ then G said be C- Neighbour Extendable graph also called strong Neighbourhood Extendable otherwise weak Neighbourhood Extendable

Definition 9.3. Let G* be tree graph,if $G_E \cong G^*$ then G said be t- Neighbour Extendable graph also called semi strong Neighbourhood Extendable

Theorem 9.4. *Let G be cycle graph and if all arcs are bondage arc , then G_E is complement of G.*

Proof. We know that G is connected graph and each node adjacent two nodes since G is cycle,so strong neighbour of each node of G is two.Clearly, if v_i, v_j, are adjacent nodes then $N_s(v_i) \cap N_s(v_j) = \emptyset$ but alternative nodes have some same nodes, so make arcs between them it will be form complement of G. □

Corollary 9.5. *Let G be cycle graph with n nodes then G is strong neighbour extendable if n is odd, otherwise weak neighbour extendable*

Proof. Case 1: If n is odd, by using theorem 9.4 there exist two paths in G_E ie one is $v_1, v_3, ...v_n, v_2$ say P_1 another path say P_2 is $v_2, v_4, ...v_{n-1}, v_1$ so P_1 and P_2 has same nodes say $v_1 and v_2$ clearly G_E is connected graph and G is strong neighbour extendable
Case 2: If n is even, by using theorem 9.4 there exist two paths in G_E ie one is $v_1, v_3, ...v_{2n-1}v_1$ say P_1 another path say P_2 is $v_2, v_4, ...v_{2n}, v_2$ so P_1 and P_2 does not have same nodes.Clearly G_E is disconnected graph and G weak neighbour extendable □

Theorem 9.6. *Let G be complete graph with n nodes, then G is strong neighbour extendable*

Proof. We know that G complete graph then every node has n-1 strong arcs so strong neighbour set of every node has n-1 nodes $N_S(v_i) = \{v_1, v_2, ...v_{i-1}, v_{i+1}, ..v_n\} \forall v_i \epsilon G$ then $N_S(v_i) \bigcap N_s(v_j) = \{v_1, v_2, ...v_{i-1}, v_{i+1}, ..v_{j-1}, v_{j+1}, ..v_n\} \forall v_i, v_j \epsilon G$,this implies that G_E is complete graph,so G is strong neighbour extendable □

Theorem 9.7. *Let P_n be path with n nodes then G is not neighbour extendable.*

Proof. LetP_n be path with n nodes then there exist unique path , so every arcs are strong arc and two nodes have one strong neighbour other nodes have two strong neighbours, but their neighbour sets are distinct, so G_E is null graph therefore G is not neighbour extendable. □

Theorem 9.8. *Let G be complete bipartite graph then G is not strong neighbour extendable*

Proof. Given G is complete bipartite graph so node set is partition of two set say V_1 and V_2 then each node of V_1 is strong neighbour of every node in V_2 there two distinct path form in G_E ,One path connect every node in V_1 another path connect every node in V_2 so G_E is disconnect graph therefore G is not strong neighbour extendable. □

Theorem 9.9. *Let G be wheel graph with p nodes then G is Strong neighbour Extendable*

Proof. Given that G is wheel graph then the domination number of G is one (say v_i) clearly v_i incident with p-1 bondage so strong neighbourhood set of every node of G has node v_i therefore G_E is connected graph implies that G is strong neighbour extendable. □

Definition 9.10. (Deficiency Number) Let G be fuzzy graph but G is not strong neighbour extendable then the deficiency number is required number of arcs to make G is strong neighbour extendable .

Theorem 9.11. *Let G be graph with 2n nodes which G is not neighbour extendable then deficiency number of G is one.*

Proof. Given G is cycle with even nodes by using corollary 9.5 there exist two distinct cycle so add one strong arc between any node from one cycle and another cycle then G_E is connected graph therefore neighbour extendable . □

Theorem 9.12. *Let G be complete bipartite graph which G is not neighbour extendable then deficiency number of G is two .*

Proof. Given G is complete bipartite graph by using theorem 9.8 there exist two open path so need two arcs for make G_E is cycle therefore G is strong neighbour extendable. □

10 Conclusion

$L^*(G)$ is not tree, if all arc of G are strong. Above non bondage value ($\neq 0$) is not true for all graphs because $K_{1,n}$ or star graph and P_3 non-bondage value is 0 and also bondage number is equal to 1 for such above graphs

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