Properties Of Universally Prestarlike Functions

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Abstract

Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathcal{C} \setminus [1, \infty)$ have been recently introduced by S. Ruscheweyh.This notion generalizes the corresponding one for functions in the unit disk Δ(and other circular domains in $\mathcal{C}$). In this paper, we obtain properties of universally prestarlike functions of order α.

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1. Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain Ω. For domain Ω containing the origin $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when Ω is the open unit disk $\Delta = \{z \in \mathcal{C} : |z| < 1\}$, we use the abbreviation H, H_0 and H_1 respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order α with $(0 \leq \alpha < 1)$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \Delta) \tag{1.1}$$

and the set of all such functions is denoted by S_α. The convolution or Hadamard Product of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$

is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in H_1$ is called prestarlike of order α if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S_\alpha \tag{1.2}$$

The set of all such functions is denoted by $\mathcal{R}_\alpha$. The notion of prestarlike functions has been extended from the unit disk to other disk and half planes containing the origin by Ruscheweyh and Salinas(see [2]). Let Ω be one such disk or half plane.Then there are two unique parameters $\gamma \in \mathcal{C} \setminus \{0\}$ and $\rho \in [0,1]$ such that

$$\Omega_{\gamma,\rho} = \{w_{\gamma,\rho}(z) : z \in \Delta\} \tag{1.3}$$

where,

$$w_{\gamma,\rho}(z) = \frac{\gamma z}{1-\rho z}.$$

Note that $1 \notin \Omega_{\gamma,\rho}$ iff $|\gamma + \rho| \leq 1$.

Definition 1.1. (see[1][2][3]) Let $\alpha \leq 1$, and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in H_1(\Omega_{\gamma,\rho})$ is called prestarlike of order α in $\Omega_{\gamma,\rho}$ if

$$f_{\gamma,\rho}(z) = \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha \tag{1.4}$$

The set of all such functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.
Let Λ be the slit domain $\mathcal{C} \setminus [1, \infty)$(the slit being along the positive real axis).

Definition 1.2.(see[1][2][3]) Let $\alpha \leq 1$. A function $f \in H_1(\Lambda)$ is called universally prestarlike of order α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. The set of all such functions is denoted by $\mathcal{R}_\alpha^u$.

Note1.1.(see[2]) Let$F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz}$ where $a_k = \int_0^1 t^k d\mu(t)$, $\mu(t)$ is a probability measure on $[0,1]$. Let T denote the set of all such functions F. They are analytic in the slit domain Λ.

Lemma 1.3.(see [6]) Let $w(u, v)$ be a complex valued function, that is

$$w : \mathcal{D} \to \mathcal{C} \qquad (\mathcal{D} \subset \mathcal{C} \times \mathcal{C})$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$
Suppose that the function $w(u, v)$ satisfies the following conditions:

1. $w(u, v)$ is continuous in $\mathcal{D}$;

2. $(1, 0) \in \mathcal{D}$ and $Re\{w(1, 0)\} > 0$;

3. $Re\{w(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in \mathcal{D}$ and such that

$$v_1 \leq -\frac{(1 + u_2^2)}{2}$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

be regular in Δ such that

$$(p(z), zp'(z)) \in \mathcal{D}$$

for all $z \in \Delta$. If

$$Re\{w(p(z), zp'(z))\} > 0$$

then

$$Re\{p(z)\} > 0.$$

Some Properties of Universally prestarlike functions are discussed in (see[4][5]).

2.Properties of Universally prestarlike functions of order α

THEOREM 2.1.If $f \in H_1(\Lambda)$ satisfies

$$\Re\left\{\frac{D^{\beta+2}f(z)}{D^{\beta+1}f(z)}\right\} > \beta_1$$

$(z \in \Delta, \beta = 2 - 2\alpha, 0 \leq \alpha < 1.)$ for some β_1 $(\frac{1}{2} \leq \beta_1 < 1)$, then

$$\Re\left\{\frac{D^{\beta+1}f(z)}{D^{\beta}f(z)}\right\} > \gamma$$

where,

$$\gamma = \frac{(2\beta_1(\beta+2)-3)+\sqrt{(2\beta_1(\beta+2)-3)^2+8(\beta+1)}}{4(\beta+1)} \tag{2.0}$$

Hence $f \in \mathcal{R}^u_\alpha$. The result is Sharp.
P r o o f. It is known that for $\beta \geq 0$

$$z(D^\beta f(z))' = (\beta+1)D^{\beta+1}f(z) - \beta D^\beta f(z) \tag{2.1}$$

where $(D^\beta f)(z) = \frac{z}{(1-z)^\beta} \star f$, for $\beta \geq 0$.In particular, for $\beta = n \in$ N. we have $D^{n+1}f = \frac{z}{n!}(z^{n-1}f)^{(n)}$. This implies

$$\frac{z(D^\beta f(z))'}{D^\beta f(z)} = (\beta+1)\frac{D^{\beta+1}f(z)}{D^\beta f(z)} - \beta \tag{2.2}$$

If we define the function $p(z)$ by

$$\frac{D^{\beta+1}f(z)}{D^\beta f(z)} = \gamma + (1-\gamma)p(z) \tag{2.3}$$

with γ defined as before (2.0), then

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

is analytic in Δ.
Now, differentiating both sides of equation (3.3) logarithmically, we have

$$(\beta+2)\frac{D^{\beta+2}f(z)}{D^{\beta+1}f(z)} = (\beta+1) + \frac{z(D^\beta f(z)')}{D^\beta f(z)} + \frac{(1-\gamma)p'(z)}{\gamma+(1-\gamma)p(z)}. \tag{2.4}$$

Now, using (2.1) in (2.4) we get,

$$\frac{D^{\beta+2}f(z)}{D^{\beta+1}f(z)} = \frac{\beta+1}{\beta+2}\frac{D^{\beta+1}f(z)}{D^\beta f(z)} + \frac{1}{\beta+2} + \frac{(1-\gamma)zp'(z)}{(\beta+2)(\gamma+(1-\gamma)p(z))} \tag{2.5}$$

which readily yields

$$Re\left\{\frac{D^{\beta+2}f(z)}{D^{\beta+1}f(z)}\right\} > \beta_1.$$

Therefore, if we define the function $w(u,v)$ by

$$w(u,v) = (\beta+1)\gamma + (\beta+1)(1-\gamma)u(z) + 1 - \beta_1(\beta+2) + \frac{(1-\gamma)v(z)}{\gamma+(1-\gamma)u(z)} \tag{2.6}$$

then we see that

1. $w(u,v)$ is continuous in $\mathcal{D} = \mathcal{C} \setminus [1,\infty)$

2. $(1,0) \in \mathcal{D}$ and $Re\{w(1,0)\} = (\beta+2)(1-\beta_1) > 0$

3. for all $(iu_2, v_1) \in \mathcal{D}$ and such that

$$v_1 \leq -\frac{(1+u_2^2}{2}$$

$$\begin{aligned} Re\{w(iu_2,v_1)\} &= (\beta+1)\gamma + 1 - \beta_1(\beta+2) + \frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)u_2^2} \\ &\leq (\beta+1)\gamma + 1 - \beta_1(\beta+2) + \frac{\gamma(1-\gamma)(1+u_2^2)}{2(\gamma^2+(1-\gamma)u_2^2)} \end{aligned}$$

Now, by simple computation and using (2.0) we get

$$2(\beta+1)\gamma^2 - (2\beta_1(\beta+2)-3)\,\gamma - 1 = 0$$

for $\beta_1 \geq \gamma$ and $\beta_1 \geq \dfrac{1}{2}$.
Hence $Re\{w(iu_2,v_1)\} \leq 0$. This implies that the function $w(u,v)$ satisfies the hypothesis of lemma 1.3. Thus we conclude that

$$\Re\left\{\frac{D^{\beta+1}f(z)}{D^{\beta}f(z)}\right\} > \gamma$$

which completes the proof. ∎

Corollary 2.2. If $\beta = 2-2\alpha \geq 0$ and $0 \leq \beta_1 < 1$, $0 \leq \alpha < 1$, then

$$\mathcal{R}^u_{\beta+1}(\beta_1) \subset \mathcal{R}^u_{\beta}\left((\beta+1)(\gamma-\beta)\right)$$

where,γ is defined as before in (2.0) and

$$(\beta+1)(\gamma-\beta) \geq \beta_1$$

.

P r o o f. Let $f \in \mathcal{R}^u_{\beta+1}(\beta_1)$. Then we have

$$Re\left\{\frac{z(D^{\beta+1}f(z))'}{D^{\beta+1}f(z)}\right\} > \beta_1 \tag{2.7}$$

By a simple computation, using (2.2) and (2.7), we obtain

$$Re\left\{\frac{D^{\beta+2}f(z)}{D^{\beta+1}f(z)}\right\} > \frac{\beta+\beta_1+1}{\beta+2} \tag{2.8}$$

Applying the theorem (2.1) we have

$$Re\left\{\frac{D^{\beta+1}f(z)}{D^{\beta}f(z)}\right\} > \gamma \tag{2.9}$$

where γ is defined as before in (2.0) Now, by a simple computation we get

$$\frac{z(D^{\beta}f(z))'}{D^{\beta}f(z)} = \frac{(\beta+1)D^{\beta+1}f(z)}{D^{\beta}f(z)} - \beta$$

This implies

$$\frac{z(D^{\beta}f(z))'}{D^{\beta}f(z)} > (\beta+1)(\gamma-\beta)$$

Hence

$$f \in \mathcal{R}^{u}_{\beta}\left((\beta+1)(\gamma-\beta)\right)$$

which completes the corollary.

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