Quasi Nonexpansive Sequences In Dislocated Quasi - Metric Spaces

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Abstract :

We introduce the notion of a quasi - nonexpansive sequence with respect to a non - empty subset of a dislocated quasi - nonexpansive metric space and extend the results of **M.A.Ahmed and F.M.Zeyada [1]** to such sequences.

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Dislocated quasi - nonexpansive w.r.to F , quasi – metric spaces , asymptotically regular , decreasing sequences and metric spaces .

1 . INTRODUCTION

M.A.Ahmed and F.M.Zeyada [1] established the convergence of a sequence $\{x_n\}$, in a dislocated - quasi metric space (X,d) if a map $T : X \longrightarrow X$ is quasi – nonexpansive with respect to $\{x_n\}$. We observe that the role playe by the map T in proving the convergence , is meagre . Consequently , we introduce the notion of a quasi – nonexpansive sequence with respect to a non empty subset of a dislocated quasi metric space and establish the convergence of such sequences under certain conditions . These results extend the results of [1] .

We begin with various definitions

Definition 1.1:

Let X be a non - empty set and let $d : X \times X \longrightarrow [0,\infty)$ be a function called a distance function satisfying one or more of (1.1.1) − (1.1.5) .

(1.1.1) : $d(x,x) = 0 \ \forall\ x \in X.$

(1.1.2) : $d(x,y) = d(y,x) = 0 \ \Rightarrow\ x = y \ \forall\ x, y \in X.$

(1.1.3) : $d(x,y) = d(y,x) \ \forall\ x, y \in X.$

(1.1.4) : $d(x,y) \le d(x,z) + d(z,y) \ \forall\ x, y, z \in X$.

(1.1.5) : $d(x,y) \le \max\{d(x,z), d(z,y)\} \ \forall\ x, y, z \in X$.

(i) If d satisfies (1.1.2) and (1.1.4) then d is called a dislocated quasi metric (or) dq - metric and (X ,d) is called a dq - metric space .

(ii) If d satisfies (1.1.2) , (1.1.3) and (1.1.4) then d is called a dislocated metric and (X ,d) is called a dislocated metric space.

(iii) If d satisfies (1.1.1) , (1.1.2) and (1.1.4) then d is called a quasi metric (or) q – metric and (X , d) is called a quasi metric space (or) q – metric space .

(iv) If d satisfies (1.1.1) , (1.1.2) , (1.1.3) and (1.1.4) then d is called a metric and (X , d) is called a metric space .

(v) If d satisfies (1.1.1) , (1.1.2) , (1.1.3) and (1.1.5) then d is called an ultra metric and (X , d) is called an ultra metric space .

We observe that every ultra metric is a metric .

Let D be a subset of a quasi metric space (X , d) and $T : D \longrightarrow X$ be any mapping . Assume that $F(T)$ is the set of all fixed points of T . For a given $x_0 \in D$, the sequence of iterates $\{x_n\}$ is defined by

(I) : $x_n = T(x_{n-1}) = T^n(x_0)$, where $n \epsilon N$ and N is the set of all positive integers .

Definition 1.2 : (F.M.Zeyada , G.H.Hassan and M.A.Ahmed [11])

A sequence $\{x_n\}$ in a dislocated quasi metric space (X , d) is called Cauchy , if to each $\epsilon > 0$, there exists $n_0 \in N$, such that for all $m, n \geq n_0$, $d(x_m , x_n) < \epsilon$.

Definition 1.3 :

A sequence $\{x_n\}$ in a dislocated quasi metric space (X , d) is said to be dislocated quasi - convergent (or) dq - convergent to x, if

$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x , x_n) = 0$.

In this case x is called a dislocated quasi – limit (or) dq - limit of $\{x_n\}$ and we write $x_n \longrightarrow x$. It can be shown that dq-limit of a sequence $\{x_n\}$, if exists is unique .

Note : In a dislocated quasi metric space , when we talk of dq – convergence or dq – limit , we conveniently drop the prefix " dq" , in the absence of any ambiguity .

Definition 1.4 :

A dislocated quasi metric space (X , d) is complete , if every Cauchy sequence in it is dq - convergent .

Definition 1.5 :

Let (X , d) be a dislocated quasi metric space . Let $\phi \neq A \subseteq X$.

Then $d(x, A) = inf_{a\in A}\{d(x, a), d(a, x)\}$.

Definition 1.6 : (M.A.Ahmed and F.M.Zeyada [1] , definition 2.1)

Let (X , d) be a quasi - metric space and $\phi \neq D \subset X$. The mapping $T : D \longrightarrow X$ is said to be quasi - nonexpansive w.r.to a sequence $\{x_n\}$ of D , if for all $n \in N \cup \{0\}$ and for every $p \in F(T)$,

$d(x_{n+1}, p) \leq d(x_n, p)$, where $F(T)$ = the fixed point set of T

(we assume that $F(T) \neq \phi$) .

The following results are proved in (F.M.Zeyada , G.H.Hassan and M.A.Ahmed [11])

Lemma 1.7 : (F.M.Zeyada , G.H.Hassan and M.A.Ahmed [11])

Let (X , d) be a dislocated quasi metric space .

Then every dq – convergent sequence in X is Cauchy .

It may be noted that the converse of **lemma 1.7** is not true .

Lemma 1.8 : (F.M.Zeyada , G.H.Hassan and M.A.Ahmed [11])

Let (X , d) be a dislocated quasi metric space . If $\{x_n\}$ is a sequence in X dq - converging to $x \in X$, then every subsequence of $\{x_n\}$ dq - converges to x .

Lemma 1.9 : (F.M.Zeyada , G.H.Hassan and M.A.Ahmed [11])

Dislocated quasi – limits in a dq – metric space are unique .

(M.A.Ahmed and F.M.Zeyada [1]) proved the following results .

Theorem 1.10 : (M.A.Ahmed and F.M.Zeyada [1] , Theorem 2.1)

Let $\{x_n\}$ be a sequence in a subset D of a q – metric space (X , d) and $T : D \longrightarrow X$ be a map such that $F(T) \neq \phi$. Then

(a) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ if $\{x_n\}$ converges to a unique point in $F(T)$;

(b) $\{x_n\}$ converges to a unique point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0$, $F(T)$ is a closed set , T is quasi - nonexpansive w.r.to $\{x_n\}$ and X is complete .

Theorem 1.11 : (M.A.Ahmed and F.M.Zeyada [1] , Theorem 2.2)

Let $\{x_n\}$ be a sequence in a subset D of a complete q – metric space (X , d) and $T : D \longrightarrow X$ be a map such that $F(T) \neq \phi$ is a closed set . Assume that

(i) T is quasi - nonexpansive w.r.to $\{x_n\}$;

(ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$

(iii) if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, then

$\liminf_n d\big(y_n, F(T)\big) = 0$ or $\limsup_n d\big(y_n, F(T)\big) = 0$.

Then $\{x_n\}$ converges to a unique point in $F(T)$.

Note : The presence of conditions (ii) and (iii) guarantees that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We show in Example 2.6 that condition(ii) alone may not guarantee that $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

2 . MAIN RESULTS

In this section , we introduce the notion of a quasi – nonexpansive sequence with respect to a non - empty subset of a dislocated quasi – nonexpansive metric space and extend the results in **[1]** to such spaces .

Definition 2.1 :

Let (X , d) be a dislocated quasi metric space , $\phi \neq F \subset X$ and $\{x_n\} \subset X$ such that $x_n \notin F \ \forall n = 1,2,3, \ldots$,

Then $\{x_n\}$ is said to be quasi - nonexpansive w.r.to F , if

$$d(x_{n+1}, p) \le d(x_n, p)$$

and

$$d(p, x_{n+1}) \le d(p, x_n) \quad \forall p \in F \text{ and } n = 1,2,3, \ldots ,$$

Lemma 2.2 :

Let (X, d) be a dislocated quasi metric space, $\phi \neq F \subseteq X$ and $\{x_n\} \subset X$, $x_n \notin F \ \forall n$. Suppose that $\{x_n\}$ is quasi - nonexpansive w.r.to F. Then $\lim_{n\to\infty} d(x_n, F) = 0 \Longrightarrow d(x_n, x_{n+1}) \longrightarrow 0$ and $d(x_{n+1}, x_n) \longrightarrow 0$.

Proof :

Let $\epsilon > 0$. Then there exists $p \in F$ and positive integer M such that

$d(x_M, p) < \frac{\epsilon}{2}$ and $d(p, x_M) < \frac{\epsilon}{2} \quad \forall n \geq M$.

$\because \{x_n\}$ is quasi - nonexpansive w.r.to F,

$$d(x_{n+1}, p) \leq d(x_n, p) \leq \cdots \leq d(x_M, p) < \frac{\epsilon}{2}$$

and

$$d(p, x_{n+1}) \leq d(p, x_n) \leq \cdots \leq d(p, x_M) < \frac{\epsilon}{2}$$

Now

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, p) + d(p, x_{n+1}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \quad \forall n \geq M. \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, p) + d(p, x_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \quad \forall n \geq M. \end{aligned}$$

$\therefore d(x_n, x_{n+1}) \longrightarrow 0$ and $d(x_{n+1}, x_n) \longrightarrow 0$.

Lemma 2.3 :

Suppose $\{x_n\}$ is quasi - nonexpansive w.r.to F. Then $\lim_{n\to\infty} d(x_n, F) = 0 \Longrightarrow \{x_n\}$ is a Cauchy sequence.

Proof :

Let $\epsilon > 0$. Then there exists a positive integer M such that

$d(x_n, F) < \frac{\epsilon}{2} \quad \forall n \geq M$.

Now

$$d(x_M, F) < \frac{\epsilon}{2} \Longrightarrow \exists\, p \in F \ni d(x_M, p) < \frac{\epsilon}{2} \text{ and } d(p, x_M) < \frac{\epsilon}{2}$$

$$\therefore d(x_n, p) \leq d(x_M, p) < \frac{\epsilon}{2}$$

and

$$d(p, x_n) \leq d(p, x_M) < \frac{\epsilon}{2} \quad \forall n \geq M.$$

Now suppose $m, n \geq M$. Then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, p) + d(p, x_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

and

$$d(x_n, x_m) \leq d(x_m, p) + d(p, x_m)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

$\therefore \{x_n\}$ is a Cauchy sequence .

Lemma 2.4 :

Let (X, d) be a dislocated quasi metric space and $\{x_n\}$ be a sequence in X . Assume that F be a non – empty subset of X . If $\{x_n\}$ is quasi - nonexpansive w.r.to F , then $d(x_n, F)$ is a monotonically decreasing sequence in $[0, \infty)$.

Proof :

Since $\{x_n\}$ is quasi - nonexpansive w.r.to F,
$d(x_{n+1}, p) \leq d(x_n, p) \rightarrow$ (2.4.1) , for all $n \in N \cup \{0\}$ and for every $p \in F$.
From (2.4.1) , taking the infimum over $p \in F$, we get that
$d(x_{n+1}, F) \leq d(x_n, F)$ for all $n \in N \cup \{0\}$.
Hence $\{d(x_n, F)\}$ is a monotonically decreasing sequence in [0,∞) .

Lemma 2.5 :

Let (X, d) be a dislocated quasi metric space and $\{x_n\}$ be a sequence in X .
Suppose $\{x_n\}$ is quasi - nonexpansive w.r.to $F \neq \phi$ satisfying $\lim_{n\rightarrow\infty} d(x_n, F) = 0$.
Then $\{x_n\}$ is a Cauchy sequence .

Proof :

Since $\{x_n\}$ is a quasi - nonexpansive w.r.to $F \neq \phi$, to each $\epsilon > 0$,
there exists $p \in F$ and positive integer M such that

$$d(x_m, p) < \frac{\epsilon}{2} \quad \text{and} \quad d(p, x_n) < \frac{\epsilon}{2} \quad \forall\ m, n \geq M .$$

Suppose $m, n \geq M$. Then

$$d(x_m, x_n) \leq d(x_m, p) + d(p, x_n)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

and

$$d(x_n, x_m) \leq d(x_m, p) + d(p, x_m)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

$\therefore \{x_n\}$ is a Cauchy sequence .

The following example shows that converse of Lemma 2.5 is not true .

Example 2.6 :

X = { (-1,0) , (1,0) and the segment [(0,1) ,(0,2)] of the Y- axis }
d is the usual Euclidean distance in R^2 .
F = { (-1, 0) , (1,0)} , x_n = (0, 1+ $\frac{1}{n}$) , $n = 1,2,3$ ...
Then $\{x_n\}$ is dislocated quasi - nonexpansive w.r.to F
$d(x_n, F) \geq d(x_{n+1}, F)$,

$d(x_n, x_{n+1}) \longrightarrow 0$ and $x_n \longrightarrow (0,1) \notin F$.

Now we state and prove our first main result , which is an extension of Theorem 1.10 to quasi – nonexpansive sequences .

Theorem 2.7 :

Let $\{x_n\}$ be a sequence in a subset D of a dislocated quasi – metric space (X, d) and $\phi \neq F \subset D$ $(x_n \notin F \; \forall \, n)$. Then

(a) $\lim_{n\to\infty} d(x_n, F) = 0$, if $\{x_n\}$ converges to a point in F

(b) $\{x_n\}$ converges to a unique point in F , if $\lim_{n\to\infty} d(x_n, F) = 0$, F is a closed set , $\{x_n\}$ is quasi - nonexpansive w.r.to F and X is complete .

Proof of (a) :

Since $\{x_n\}$ converges to a point in F , there exists a point $p \in F$ such that $\lim_{n\to\infty} d(x_n, p) = 0$ and $\lim_{n\to\infty} d(p, x_n) = 0$

$\therefore$ Given $\epsilon > 0$, there exists a positive integer M such that

$d(x_n, p) < \frac{\epsilon}{2}$ and $d(p, x_n) < \frac{\epsilon}{2}$ $for\ every\ n \geq M$.

$$\therefore \; d(p,p) \leq d(p, x_n) + d(x_n, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall\, n \geq M .$$

$\therefore \; d(p,p) < \epsilon$ for every $\epsilon > 0$,

$\therefore \; d(p,p) = 0$.

Now

$d(x_n, F) \leq d(x_n, p) < \frac{\epsilon}{2}$ $\forall \; n \geq M$.

$\therefore$ $\lim_{n\to\infty} d(x_n, F) = 0$

$\therefore$ (a) holds .

Proof of (b) :

Let (X, d) be a complete dislocated quasi - metric space and $\{x_n\}$ be a sequence in X and $\phi \neq F \subset X$. Assume that $\{x_n\}$ is quasi - nonexpansive w.r.to F , F is closed and $\lim_{n\to\infty} d(x_n, F) = 0$. Then $\{x_n\}$ is a Cauchy sequence by lemma 2.5 , hence there exists p such that $\{x_n\}$ converges to p .

Let $\epsilon > 0$. There exists a positive integer M such that

$d(x_n, F) < \frac{\epsilon}{2}$ for every $n \geq M$ and

$d(x_n, p) < \frac{\epsilon}{2}$ and $d(p, x_n) < \frac{\epsilon}{2}$ $\forall\, n \geq M$.

$\therefore$ There exists $q_M \in F$ such that

$d(x_M, q_M) < \frac{\epsilon}{2}$ and $d(q_M, x_M) < \frac{\epsilon}{2}$

$$\therefore \; d(p, q_M) \leq d(p, x_M) + d(x_M, q_M) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$d(p, q_M) < \epsilon$

and similarly we have

$d(x_{n+1}, F) \leq d(x_{n+1}, P) \leq d(x_n, p) \; \forall \; p \in F$

$\Rightarrow \quad d(q_M, p) \leq d(q_M, x_M) + d(x_M, p)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore \quad d(q_M, p) < \epsilon$

$\therefore$ p is a limit point of F

$\therefore$ $p \in F$, since F is closed

Since limits are unique (by **lemma 1.9) ,** $\{x_n\}$ converges to a unique point $p \in F$.

Hence (b) holds .

The following theorem which is an analogue of Theorem 1.11 establishes the convergence of the sequence .

Theorem 2.8 :

Let (X , d) be a complete dislocated - quasi metric space . Assume that $\{x_n\}$ is a sequence in X and $\phi \neq F \subseteq X$. Further assume that there is a mapping $\varphi : [0,\infty) \to [0,1)$ such that φ is monotonically increasing and

$d(x_{n+1}, F) \leq \varphi(d(x_n, F))\, d(x_n, F)$ for $n = 1,2,3 \ldots \longrightarrow$ (2.8.1)

Then $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to a point q . If further F is closed then $q \in F$.

Proof :

By hypothesis

$d(x_{n+1}, F) \leq \varphi(d(x_n, F))\, d(x_n, F) \leq d(x_n, F)$,

so that $\{ d(x_n, F)\}$ is decreasing and hence $\{ \varphi(d(x_n, F))\}$ is decreasing since φ is increasing .

$$\begin{aligned}\therefore\ d(x_{n+1}, F) &\leq \varphi(d(x_n, F)) d(x_n, F) \leq \varphi(d(x_n, F))\, \varphi(d(x_{n-1}, F)) d(x_{n-1}, F) \\ &\leq \varphi(d(x_n, F))\varphi(d(x_{n-1}, F)) \ldots\ldots\ \varphi(d(x_1, F)) d(x_1, F) \\ &\leq \varphi(d(x_1, F))\varphi(d(x_1, F)) \ \ldots\ldots\ \ldots \varphi(d(x_1, F)) d(x_1, F) \\ &= (\varphi(d(x_1, F)))^n . d(x_1, F) \to 0 \text{ as } n \to \infty . \\ &\quad (\because \varphi(d(x_n, F) < 1)\end{aligned}$$

Thus $d(x_n, F) \to 0$ as $n \to \infty$.

$\therefore$ By lemma 2.5 , $\{x_n\}$ is Cauchy sequence and hence converges to a point q since X is complete . If F is closed by (Theorem (2.7) (b)) follows that $q \in F$.

The following Example shows that

Theorem 2.8 may not hold good if (2.8.1) is replaced by

$d(x_{n+1}, F) \leq d(x_n, F)$ for $n = 1,2,3, \ldots \longrightarrow$ (2.8.2)

even if we assume that (even in a metric space)

$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \longrightarrow$ (2.8.3)

Example 2.9 :

Let X be the subset of R x R consisting of the points (-1,0) , (1,0) and the segments of the $Y -$ axis joining the two points (0,1) and (0,2) .

Hence $X = \{(-1,0), (1,0)$ and $\{(0, y)/ 1 \leq y \leq 2 \}\}$

Let d be the Euclidean metric in R^2 .

Take $F = \{ (-1,0), (1,0)\}$ and $x_n = \{(0, 1 + \frac{1}{n})/ n = 1,2, . . \}$

Then $\{x_n\}$ is quasi – nonexpansive w.r.to to F ,

$d(x_{n+1}, F) \leq d(x_n, F)$ for $n = 1,2,3, \ldots$

$d(x_n, x_{n+1}) \longrightarrow 0$, F is closed but $\{d(x_n, F)\}$ does not converges to 0.

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