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Reformulation on Modified Runge-Kutta Third Order Methods for Solving Initial Value Problems

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***Abstract* – Ordinary Differential equation with Initial Value Problems (IVP) frequently arise in many physical problems. Numerical methods are widely used for solving the problems especially in case of numerical simulation. Several numerical methods are available in the literature for solving IVP. Runge-Kutta (which is actually Arithmetic Mean (AM) based method) is one of the best commonly used numerical approaches for solving the IVP. Recently Evans[1] proposed Geometric Mean (GM) based Runge-Kutta third order method and Wazwaz [2] proposed Harmonic Mean (HM) based Runge-Kutta third order method for solving IVP. Also Yanti et al. [3] proposed the linear combination of AM, HM and GM based Runge-Kutta third order method. We extensively perform several experiments on those approaches to find robustness of the approaches. Theoretically as well as experimentally we observe that GM based and Linear combination of AM, GM and HM based approaches are not applicable for all kinds of problems. To overcome some of these drawbacks we propose modified formulas correspond to those AM and linear combination of AM, GM, HM based methods. Experimentally it is shown that the proposed modified methods are most robust and able to solve the IVP efficiently.**

Keywords—Initial value problem; Runge-Kutta method; Arithmetic mean; Harmonic mean, Geeometric mean;

I. INTRODUCTION

Many problems in science and engineering when formulated mathematically are readily expressed in terms of linear or non linear ordinary differential equations with appropriate initial or boundary conditions. For example, the trajectory of a ballistic missile, the motion of an artificial satellite in its orbit is governed by ordinary differential equations. Theories concerning electrical networks, bending of beams, stability of aircraft etc., are modeled by differential equations. To be more precise, the rate of change of any quantity with respect to another can be modeled by an ordinary differential equation. In some of the cases analytical approach is not effective and some cases numerical approach is only possible one. Among the existing numerical methods Runge-Kutta is widely used computationally efficient methods in terms of accuracy. There exist several versions of Runge-Kutta methods, namely second order, third order, fourth order and so on. The classical third order Range-Kutta method can be expressed as Arithmetic Mean (AM) based approach [1]. Recently Evans [1] proposed modified Runge-Kutta third order method by using Geometric Mean (GM) instead of AM. He performed an experiment and showed that GM based Runge-Kutta third order approach is comparable with existing AM based Runge-Kutta third order method. On the other hand Wazwaz [2] reassessed Runge-Kutta third order by using Harmonic Mean (HM). He showed that HM based Runge-Kutta third order approach [2] performed better than GM based Runge-Kutta third order approach. Very recently Yanti et al. [3] proposed a linear combination of Arithmetic Mean, Geometric Mean and Harmonic Mean based Runge-Kutta third order approach. Experimentally they showed that their proposed method performed better than AM based approach in many cases and comparable with GM based as well as HM based approaches.

II. EXIXTING METHODS

Suppose a first order Initial Value Problem (IVP) is of the following form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{1}$$

The autonomous structure of "(1)"is as follows [4]:

$$y'(x) = f(y(x)), \quad y(x_0) = y_0 \tag{2}$$

For solving "(1)", Evans [1] define the classical Runge-Kutta third order method as follows

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\ k_3 &= f(x_n + h, y_n - hk_1 + hk_2) \end{aligned}\right\} \tag{3}$$

and

$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 2k_2 + k_3) \tag{4}$$

Evan [1], Hall et al. [5] and Jacquez [6] rewrite "(4)" as follows:

$$y_{n+1} = y_n + \frac{h}{2}(\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2}) \tag{5}$$

Here $(k_1 + k_2)/2$ and $(k_2 + k_3)/2$ are arithmetic mean, so "(3)" and "(5)" are known as Runge-Kutta method based on Arithmetic Mean (RKAM) [1].

Evans [1] proposed a modified RKAM method based on Geometric Mean (RKGM) instead of arithmetic mean as follows:

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1) \\ k_3 &= f(x_n + \frac{2h}{3}, y_n - \frac{h}{2}k_1 + \frac{7h}{6}k_2) \end{aligned}\right\} \tag{6}$$

and

$$y_{n+1} = y_n + \frac{h}{2}(\sqrt{k_1k_2} + \sqrt{k_2k_3}) \tag{7}$$

Wazwaz [2] also proposed a modified RKAM method based on Harmonic Mean (RKHM) instead of arithmetic mean. According to his proposed (RKHM) approach the above equations "(3)" and "(5)" are reformed as follows:

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1) \\ k_3 &= f(x_n + \frac{2h}{3}, y_n - \frac{2h}{3}k_1 + \frac{4h}{3}k_2) \end{aligned}\right\} \tag{8}$$

and

$$y_{n+1} = y_n + h(\frac{k_1k_2}{k_1+k_2} + \frac{k_2k_3}{k_2+k_3}) \tag{9}$$

On the other hand by considering RKAM, RKGM, RKHM approaches, Yanti et al. [3] proposed a third order Runge-Kutta method based on a Linear Combination of Arithmetic Mean, Harmonic Mean and Geometric Mean (RKLCM) which is as follows:

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1) \\ k_3 &= f(x_n + \frac{2h}{3}, y_n - \frac{4h}{9}k_1 + \frac{10h}{9}k_2) \end{aligned}\right\} \tag{10}$$

and

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{90}[7(k_1 + 2k_2 + k_3) \\ &- (\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}) + 32(\sqrt{k_1k_2} + \sqrt{k_2k_3})] \end{aligned} \tag{11}$$

III. PROPOSED REFORMULATION ON MODIFIED METHODS

By performing experiments (given in section IV) we observed that RKGM and RKLCM methods are not efficient at all when the slope of y i.e. $y'(x)$ of "(1)" be negative in sign. To overcome this shortcoming we proposed modified formulas. In the case of negative sign of $y'(x)$, the "(7)" of RKGM is reformulated as follows:

$$y_{n+1} = y_n - \frac{h}{2}(\sqrt{k_1k_2} + \sqrt{k_2k_3}) \tag{12}$$

"Equation (12)" including "(6)" is denoted as Modified Runge-Kutta Geometric Mean based Method-1 (MRKGM1).In the case of negative sign of $y'(x)$ of "(1)", the "(11)" of RKLCM is reformulated as follows:

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{90}[7(k_1 + 2k_2 + k_3) \\ &- (\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}) - 32(\sqrt{k_1k_2} + \sqrt{k_2k_3})] \end{aligned} \tag{13}$$

"Equation (13)" with the remaining "(10)" of RKLCM is denoted as Modified Runge-Kutta Linear Combination of AM, GM and HM based Method-1 (MRKLCM1). From the numerical experiments (see in section IV) we observed that RKGM and RKLCM methods are failed to give real-valued solution for the Prob. 4. For this shortcoming, we have tried to find out the constraint of the formulas. We observe that there exist square-root term in "(7)" and "(11)". So the value of the square-root term must be non negative. To overcome this constraint of the RKGM and RKLCM approaches we have proposed "(14)" instead of "(7)" of RKGM and "(15)" instead of "(11)" of RKLCM respectively.

The proposed reformulation equations are given below.

$$y_{n+1} = y_n + \frac{h}{2}(\sqrt{|k_1||k_2|} + \sqrt{|k_2||k_3|}) \tag{14}$$

and

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{90}[7(k_1 + 2k_2 + k_3) \\ &- (\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}) + 32(\sqrt{|k_1||k_2|} + \sqrt{|k_2||k_3|})] \end{aligned} \tag{15}$$

"Equation (14)" with "(6)" is denoted as Modified Runge-Kutta third order Geometric Mean based Method-2 (MRKGM2). "(15)" with "(6)" is denoted as Modified third order Runge-Kutta Linear Combination of AM, GM and HM based Method-2 (MRKLCM2).

IV. NUMERICAL EXPERIMENT AND DISCUSSIONS

For experimental study we consider the following four IVP problems.

Prob. 1: $\frac{dy}{dx} = \frac{1}{y}$ with initial condition $y(0) = 1$ on the interval $[0,1]$ with step size h = 0.1. Here exact solution is $y = \sqrt{2x+1}$.

Prob. 2: $\frac{dy}{dx} = y - x^2 + 1$ with initial condition $y(0) = 0.5$ on the interval $[0,2]$ with step size h = 0.2. Here exact solution is $y = (x^2 + 2x + 1) - 0.5e^x$.

Prob. 3: $\frac{dy}{dx} = -y$ with initial condition $y(0) = 1$ on the interval $[0,1]$ with step size h = 0.1. Here exact solution is $y = e^{-x}$.

Prob. 4: $\frac{dy}{dx} = -(2x + y)$ with initial condition $y(0) = -1$ on the interval $[0,0.5]$ with step size h = 0.1. Here exact solution is $y = -2x + 2 - 3e^{-x}$.

At first we have performed experiments on those IVP (Prob. 1 Prob. 2, Prob. 3 and Prob. 4) to verify the robustness of the existing modified approaches, namely RKAM, RKHM, RKGM and RKLCM. The experimental results are shown in the Table I.

Table I. Numerical Solution of **Prob. 1**, **Prob.** 2, **Prob**. 3 and **Prob. 4** with existing methods

IVP	**h**	**x**	**Exact value**	**RKAM(error)**	**RKHM(error)**	**RKGM(error)**	**RKLCM(error)**
Prob. 1	0.1	1.0	1.7320508957	*4.8378143e-003*	1.3113022e-006	5.7220459e-006	7.0333481e-006
Prob. 2	0.2	2.0	5.3054723740	*1.1412470e-001*	3.7288666e-004	1.6474724e-003	2.0971298e-003
Prob. 3	0.1	1.0	0.3678793907	*2.6882892e-002*	4.5001507e-006	*2.1140916e+000*	*1.1147479e+000*
Prob. 4	0.1	0.5	-0.8195919991	**4.9441941e-003**	1.5523434e-003	*Impossible to solve*	*Impossible to solve*

In the case of Prob. 1 and Prob. 2, we observe that all the modified methods outperform compare to classical RKAM method. In the case of Prob. 3, RKGM [1] and RKLCM [3] approaches perform worse compare to classical RKAM method. It is worthwhile to mention here that the slope of y for Prob. 1 and Prob. 2 are positive on the other hand slope of y for Prob. 3 is negative.

Proposed MRKGM1 and MRKLCM1 approaches are able to overcome this drawback. To investigate the performance of these two approaches we have performed further experiments on Prob. 3. The experimental results are summarized in the Table II. We observe in the Table II that the error of RKGM is *2.1140916e+000* and the error of RKLCM is *1.11e+00*. Whereas the error of proposed MRKGM1 is 1.1026859e-005 and the error of proposed MRKLCM1 is 1.30e-05. We also observe in the Table II that in all steps the proposed methods able to obtain much better solutions compare to RKGM and RKLCM. Moreover we observe that the error of RKAM is *2.6882892e-002* (see Table I). From this experimental study we may conclude that proposed MRKGM1 and MRAKLCM1 methods able to overcome the drawback of RKGM and RAKLCM respectively and outperform classical RKAM.

In case of Prob. 4 we observe in the Table I. that RKGM and RKLCM approaches are not able to find any solution. In order to find out the reason of failure of those methods we again have performed extensive experiments on Prob. 4. Table III displays the experimental results.

Table II. Comparison of existing and proposed methods for **Prob. 3**

x	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Exact value	0.7408182025	0.6703200340	0.6065306664	0.5488116145	0.4965852797	0.4493289292	0.4065696299	0.3678793907
RKGM	1.3135269880	1.4385291338	1.5754271746	1.7253531218	1.8895468712	2.0693662167	2.2662980556	2.4819710255
RKGM(error)	5.7270879e-001	7.6820910e-001	9.6889651e-001	1.1765416e+000	1.3929616e+000	1.6200373e+000	1.8597285e+000	*2.1140916e+000*
MRKGM1	0.7408115268	0.6703119874	0.6065215468	0.5488017201	0.4965748489	0.4493181407	0.4065586329	0.3678683639
MRKGM1(error)	**6.6757202e-006**	**8.0466270e-006**	**9.1195107e-006**	**9.8943710e-006**	**1.0430813e-005**	**1.0788441e-005**	**1.0997057e-005**	**1.1026859e-005**
RKLCM(error)	3.85e-01	5.00e-01	6.11e-01	7.18e-01	8.21e-01	9.21e-01	1.02e+00	*1.11e+00*
MRKLCM1(error)	**7.87e-06**	**9.48e-06**	**1.07e-05**	**1.16e-05**	**1.23e-05**	**1.27e-05**	**1.29e-05**	**1.30e-05**

Table III. The characteristics of RKGM and RKLCM methods to solve **Prob. 4**

Step	1	2	3	4	5
x	0.10	0.20	0.30	0.40	0.5
k_1	1	.7146995	0.45661	.2232147	**+ 0.0125855**
k_2	0.8	0.5337195	0.292836	0.0750005	**− 0.1215869**
k_3	0.8233333	0.5548339	0.3119429	0.0922921	- 0.1059334
Exact value	-0.914512277	-0.856192231	-0.822454631	-0.810960114	-0.819591999
RKGM	-0.914699495	-0.85661	-0.823214769	-0.812585473	Impossible to solve
RKGM(error)	1.87e-04	4.18e-04	7.60e-04	1.63e-03	Impossible to solve
RKLCM(error)	1.21e-04	2.71e-04	4.96e-04	1.07e-03	Impossible to solve

We observe in step five of the Table III that the sign of k_1 is positive whereas the sign of k_2 is negative. So the value of the square root of product of k_1 and k_2 is imaginary. That is why the existing geometric mean based approaches do not able to solve the Prob. 4.

Now we have again carried out some experiments on Prob. 4 to test the performance of newly proposed MRKGM2 and MRKLCM2 approaches regarding defeating of imaginary situation. The experimental results are displayed in the Table IV. In the table IV we observe that MRKGM2 and MRKLCM2 approaches are able to solve Prob. 4 successfully whereas RKGM and RKLCM are unsuccessful as well.

V. CONCLUSION

From computational experiments we have observed that the existing modified RKGM, RKHM and RKLCM methods solve the Prob. 1, and Prob. 2 efficiently compare to classical RKAM. But in the case of Prob. 3 (for the existing of negative slope) the performance of both RKHM and RKLCM is poor while our proposed both MRKGM1 and MRKLCM1 approaches show very good performance. It is worthwhile to mention here that both MRKGM1 and MRKLCM1 outper-

Table IV. Comparison of existing and proposed methods for **Prob. 4**

x	0.1	0.2	0.3	0.4	0.5
Exact value	-0.914512277	-0.856192231	-0.822454631	-0.810960114	-0.819591999
RKGM	-0.914699495	-0.85661	-0.823214769	-0.812585473	impossible to solve
RKGM(error)	1.87e-04	4.18e-04	7.60e-04	1.63e-03	impossible to solve
MRKGM2	-0.914699495	-0.85661	-0.823214769	-0.812585473	-0.804955065
M RKGM2(error)	1.87e-04	4.18e-04	7.60e-04	1.63e-03	**1.46e-02**
RKLCM(error)	1.21e-04	2.71e-04	4.96e-04	1.07e-03	impossible to solve
MRKLCM2(error)	1.21e-04	2.71e-04	4.96e-04	1.07e-03	**1.04e-02**

form all the existing RKAM, RKGM and RKLCM for solving Prob. 3. Also proposed both MRKGM1 and MRKLCM1 are comparable with RKHM for solving Prob. 3. Again in the case of Prob. 4 (for existing of negative sign of k_1k_2) both RKGM and RKLCM approaches totally failed to solve the problem whereas our proposed both MRKGM2 and MRKLCM2 approaches successfully able to solve the problem. Moreover the performance of these approaches is comparable to RKAM and RKHM approaches for solving Prob. 4. It is noted that the harmonic mean based existing RKHM and RKLCM approaches should not able to solve the problems in the case where $k_1 + k_2 \to 0$ and /or $k_2 + k_3 \to 0$ as harmonic mean become undefined. Finally from the experimental study we may conclude that the proposed reformulated method outperform the corresponding existing methods and relatively much more robust to solve the IVP problem.

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