#RG – Homeomorphisms in Topological Spaces

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Abstract

A bijection f: $(X,\tau) \rightarrow (Y,\sigma)$ is called #regular generalized –homeomorphism if f and f^{-1} are #rg-continuous. Also we introduce new class of maps, namely #rgc-homeomorphisms which form a subclass of #rg-homeomorphisms. This class of maps is closed under composition of maps. We prove that the set of all #rgc-homeomorphisms forms a group under the operation composition of maps.

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1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map f: X →Y when both f and f^{-1} are continuous. It is well known that as Jӓnich [[9], p.13] says correctly: homeomorphisms play the same role in topology that linear isomorphism play in linear algebra, or that biholomorphic maps play in function theory, or group isomorphism in group theory, or isometries in Riemannian geometry. In the course of generalizations of the notion of homeomorphism, Maki et al. [12] introduced g -homeomorphisms and gc -homeomorphisms in topological spaces.

In this paper, we introduce the concept of #rg-homeomorphism and study the relationship between homeomorphisms, g-homeomorphism, gs- homeomorphism and rg-homeomorphism. Also we introduce new class of maps #rgc-homeomorphism which form a subclass of #rg- homeomorphism. This class of maps is closed under composition of maps. We prove that the set of all #rgc-homeomorphisms forms a group under the operation composition of maps.

Let us recall the following definition which we shall require later.

Definition 1.1. A subset A of a space X is called

1) a preopen set[13] if A $\subseteq$ intcl(A) and a preclosed set if clint (A)$\subseteq$ A.

2) a semiopen set[10] if A $\subseteq$ clint(A) and a semi closed set if intcl (A) $\subseteq$A.

3) a regular open set[16]if A = intcl(A) and a regular closed set if A = clint(A).

4) a π- open set[20] if A is a finite union of regular open sets.

5) regular semi open[4]if there is a regular open U such U $\subseteq$A$\subseteq$cl(U).

Definition 1.2. A subset A of (X,τ) is called

1) generalized closed set (briefly, g-closed)[11] if cl (A)$\subseteq$ U whenever A$\subseteq$U and U is open in X.

2) regular generalized closed set (briefly, rg-closed)[15] if cl (A)$\subseteq$U whenever A$\subseteq$U and U is regular open in X.

3) generalized preregular closed set (briefly, gpr-closed)[8] if pcl (A)$\subseteq$U whenever A $\subseteq$ U and U is regular open in X.

4)regular weakly generalized closed set (briefly, rwg-closed)[14] if clint(A)$\subseteq$U whenever A $\subseteq$U and U is regular open in X.

5) rw-closed [3] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is regular semi open.

6)#rg-closed[17] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is rw-open.

The complements of the above mentioned closed sets are their respective open sets.

Definition: 1.3. A map f:$(X,\tau)\rightarrow(Y,\sigma)$ is called g-continuous[2](resp.rg-continuous[15] resp. gs-continuous[6], resp. gsp-continuous [7], resp. gpr-continuous[8], resp. regular continuous[15], resp. rwg-continuous[14], resp. #rg-continuous[19]) if $f^{-1}(V)$ is g-closed(resp. rg-closed, resp. gs-closed, resp. gsp-closed, resp. gpr-closed, resp. regular closed, resp. rwg-closed, resp. #rg-closed) in X for every closed subset V of Y.

Definition: 1.4. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be

(i) Irresolute[5] if $f^{-1}(V)$ is semi open in (X,τ) for each semi open set V of (Y,σ).

(iii) #rg-irresolute [19] if $f^{-1}(V)$ is #rg-open in (X,τ) for each #rg-open set V of (Y,σ).

Definition 1.5. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be

(i) #rg-closed[19] if f(F) is #rg-closed in (Y,σ) for every #rg-closed set F of (X,τ)

(ii) #rg-open[19] if f(F) is #rg-open in (Y,σ) for every #rg-open set F of (X,τ).

Definition 1.6. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be

(i) g homeomorphism[12] if both f and f^{-1} are g-continuous,

(ii) gs- homeomorphism [6] if both f and f^{-1} are gs-continuous,

(iii)rwg- homeomorphism[14] if both f and f^{-1} are rwg-continuous,

(iv) gc- homeomorphism[12] if both f and f^{-1} are gc-irresolute.

2. #RG-homeomorphism in Topological Spaces

Definition 2.1. A bijection $f:(X,\tau)\to(Y,\sigma)$ is called #regular generalized homeomorphism (briefly, #rg-homeomorphism) if f and f^{-1} are #rg-continuous.

We denote the family of all #rg-homeomorphisms of a topological space (X,τ) onto itself by #rg-h(X,τ).

Example 2.1. Consider X=Y={a,b,c,d} with topologies τ={X,ϕ, {a}, {b}, {a,b},{a,b,c}} and σ= {Y, ϕ, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is bijective, #rg-continuous and f^{-1} is #rg-continuous. Hence f is #rg-homeomorphism.

Theorem 2.1. Every homeomorphism is #rg-homeomorphism, but not conversely.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a homeomorphism. Then f and f^{-1} are continuous and f is bijection. Since every continuous function is #rg-continuous, f and f^{-1} is #rg-continuous. Hence f is #rg-homeomorphism.

The converse of the above theorem need not be true, as seen from the following example.

Example 2.2. Consider X=Y={a,b,c,d} with topologies τ= {X, ϕ, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}} and σ = {X,ϕ, {a}, {b}, {a,b},{a,b,c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is #rg-homeomorphism it is not homeomorphism, since the inverse image of closed set of {a,d} in X is {a,d} which is not closed in Y.

Theorem 2.2. Every #rg-homeomorphism is g-homeomorphism, but not conversely.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a #rg–homeomorphism. Then f and f^{-1} are #rg-continuous and f is bijection. Since every #rg-continuous function is g-continuous, f and f^{-1} are g-continuous. Hence f is g-homeomorphism.

The converse of the above theorem need not be true, as seen from the following example.

Example 2.3. Consider X=Y={a,b,c,d} with topologies τ= {X, ϕ, {a}, {b}, {a ,b}} and σ = {X,ϕ, {a}, {b}, {a ,b},{a ,b ,c}}. Define a map $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is bijection, g-continuous and f^{-1} is g-continuous. Hence f is g-homeomorphism. But f is not #rg-homeomorphism, since the inverse image of closed set of {d} in Y is {d} which is not #rg-closed in X.

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Corollary 2.1. Every #rg-homeomorphism is gs-homeomorphism, but not conversely.

Proof. By the fact that every g-homeomorphism is gs-homeomorphism and by theorem 2.2.

Corollary 2.2. Every #rg-homeomorphism is gsp-homeomorphism, but not conversely.

Proof. By the fact that every gs-homeomorphism is gsp-homeomorphism and by corollary 2.1.

Theorem 2.3. Every #rg-homeomorphism is rg-homeomorphism, but not conversely.

Proof. Let f:(X,τ) →(Y,σ) be a #rg–homeomorphism. Then f and f^{-1} are #rg-continuous and f is bijection. Since every #rg-continuous function is rg-continuous, f and f^{-1} are rg-continuous. Hence f is rg-homeomorphism.

The converse of the above theorem need not be true, as seen from the following example.

Example 2.4. Consider X=Y={a,b,c,d} with topologies τ= {X, ϕ, {a}, {b}, {a,b}} and σ = {X,ϕ, {a}, {b}, {a,b},{a,b,c}}. Define a map f:(X,τ) →(Y,σ) be the identity map. Then f is bijection, rg-continuous and f^{-1} is rg-continuous. Hence f is rg-homeomorphism. But f is not #rg-homeomorphism, since the inverse image of closed set of {d} in Y is {d} which is not # rg-closed in X.

Corollary 2.3. Every #rg-homeomorphism is rwg-homeomorphism and gpr-homeomorphism, but not conversely.

Proof. By the fact that every rg-homeomorphism is rwg-homeomorphism and gpr-homeomorphism, and by theorem 2.3.

Theorem 2.4. Let f : (X, τ) → (Y,σ) be a bijective #rg- continuous map. Then the following are equivalent.

(i) f is a #rg- open map
(ii) f is #rg-homeomorphism,
(iii) f is a #rg – closed map.

Proof. Suppose (i) holds. Let V be open in (X,τ). Then by (i), f (V) is #rg-open in (Y,σ). But f(V) = $(f^{-1})^{-1}$(V) and so $(f^{-1})^{-1}$(V) is #rg-open in (Y,σ). This shows that f^{-1} is #rg-continuous and it proves (ii).

Suppose (ii) holds. Let F be a closed set in (X,τ) . By (ii), f^{-1} is #rg-continuous and so $(f^{-1})^{-1}$(V) (F) = f(F) is #rg-closed in (Y,σ). This proves (iii).

Suppose (iii) holds. Let V be open in (X,τ). Then V^c is closed in (X,τ). By (iii), $f(V^c)$ is #rg-closed in (Y,σ). But $f(V^c)= (f(V))^c$. This implies that $(f(V))^c$ is #rg-closed in (Y,σ) and so f (V) is is #rg-open in (Y,σ). This proves (i).

Remark 2.1. The composition of two #rg-homeomorphism need not be a #rg-homeomorphism in general as seen from the following example.

Example 2.5. Consider X=Y=Z={a,b,c,d},τ= {X, ϕ, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}} and σ = {X,ϕ, {a}, {b}, {a,b},{a,b,c}} and η={Z,ϕ,{c},{a,b},{a,b,c}}. Define a map f:(X,τ)→(Y,σ) be the identity map. Then clearly f is #rg-homeomorphism. g: (Y,σ) → (Z, η) defined by g(a)=b, g(b)=c, g(c)=a and g(d)=d. Then g is also #rg – homeomorphism, but their composition gof :(X,τ)→(Z,η) is not #rg-homeomorphism, because for the closed set {a,d } of (X, τ), gof({a,d})=g(f({a,d}))=g({a,d})={b,d}, which is not #rg-closed in (Z,η). Therefore gof is not #rg-closed and so gof is not #rg-homeomorphism.

Definition 2.2. A bijection f: (X,τ) → (Y,σ) is said to be #rgc-homeomorphism if both f and f^{-1} are #rg-irresolute.

We say that spaces (X,τ) and (Y,σ) are #rgc-homeomorphic if there exists a #rgc-homeomorphism form (X,τ) onto (Y,σ).We denote the family of all #rgc–homeomorphisms of a topological space (X,τ) onto itself by #rgc-h(X,τ).

Theorem 2.5. Every #rgc-homeomorphism is a #rg – homeomorphism but not conversely.

Proof. Let f : (X,τ) → (Y,σ) be an #rgc-homeomorphism. Then f and f^{-1} are #rg – irresolute and f is bijection. By Theorem 4.2 in [19], f and f^{-1} are #rg- continuous. Hence f is #rg-homeomorphism.

The converse of the above theorem is not true in general as seen from the following example.

Example 2.6. Consider X = Y = {a, b, c,d } with τ = { X,ϕ,{a},{b},{a,b} ,{a,b,c}} and σ = {Y, ϕ, {c},{a,b}, {a,b,c}}. Let f : (X,τ) → (Y,σ) be defined by f(a)=b, f(b)=c, f(c)=a and f(d)=d. Then f is #rg-homeomorphism but it is not #rgc-homeomorphism, since f is not #rg-irresolute.

Theorem 2.6. Every #rgc-homeomorphism is g-homeomorphism but not conversely.

Proof. Proof follows from Theorems 2.5 and Theorem 2.2.

Remark 2.2 #rgc-homeomorphism and gc - homeomorphism are independent as seen from the following example.

Example 2.7 Let X = Y = {a,b,c,d} with τ = {X, ϕ, {a},{b},{a,b},{a,b,c}} and σ = {Y,ϕ,{a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}. Let f : (X, τ) $\rightarrow$ (Y, σ) be the identity map. Then f is #rgc - homeomorphism but it is not gc-homeomorphism, since f is not gc-irresolute.

Example 2.8 Consider X = Y={a,b,c,d} with topologies τ={X,ϕ,{a},{b},{a,b},{a,b,c} } and σ = {Y,ϕ,{c},{a,b},{a,b,c}}. Let f : (X, τ) $\rightarrow$ (Y,σ) be the identity map. Then f is gc – homeomorphism but it is not #rgc-homeomorphism, since f is not #rg–irresolute.

Remarks 2.3. From the above discussions and known results we have the following implications. In the following figure, by A $\rightarrow$ B we mean A implies B but not conversely and A $\not\leftrightarrow$ B means A and B are independent of each other.

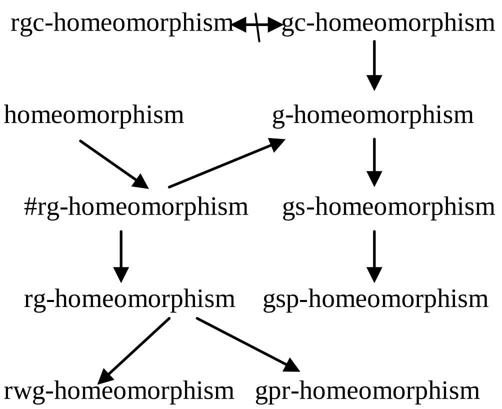

Figure 1.

Theorem 2.7. Let f : (X, τ)$\rightarrow$ (Y, σ) and g : (Y, σ) $\rightarrow$ (Z, η) are #rgc- homeomorphisms, then their composition gof: (X, τ) $\rightarrow$ (Z, η) is also #rgc-homeomorphism.

Proof. Let U be a #rg-closed set in (Z,η). Since g is #rg-homeomorphism, g^{-1} (U) is #rg-closed in (Y,σ).Since f is #rg-homeomorphism, $f^{-1}(g^{-1}(U)) = (gof)^{-1}$ (U) is #rg closed in (X,τ). Therefore gof is #rg-irresolute. Also for a #rg-closed set G in (X,τ), We have (gof)(G) = g(f(G)) = g(W), where W= f(G) . By hypothesis, f(G) is #rg-closed in (Y,σ) and so again by hypothesis, g(f(G)) is a #rg-closed set in (Z,η). That is (gof)(G) is a #rg-closed set in (Z, η) and therefore $(gof)^{-1}$ is #rg-irresolute. Also gof is a bijection. Hence gof is #rg-homeomorphism.

Theorem 2.8. The set #rgc-h(X, τ) is a group under the composition of maps.

Proof. Define a binary operation$*$: #rgc-h(X,τ) $\times$ #rgc-h(X,τ)$\rightarrow$ #rgc-h(X,τ) by f*g = gof for all f,g$\in$#rgc-h(X,τ) and o is the usual operation of composition of maps. Then by theorem 2.7, gof $\in$#rgc-h(X,τ). We know that the composition of maps is associative and the identity map I: (X,τ) $\rightarrow$ (X,τ) belonging to #rgc-h(X,τ) serves as the identity element. If f $\in$#rgc-h(X,τ), then f^{-1} $\in$ #rgc-h(X,τ) such that $fof^{-1} = f^{-1}of$ =I and so inverse exists for each element of #rgc-h(X,τ). Therefore (#rgc-h(X,τ),o) is a group under the operation of composition of maps.

Theorem 2.9. Let f: (X,τ) $\rightarrow$ (Y,σ) be a #rgc-homeomorphism. Then f induces an isomorphism from the group #rgc-h(X,τ) onto the group #rgc-h(Y,σ).

Proof. Using the map f, we define a map Ψ_f: #rgc-h (X,τ)$\rightarrow$ #rgc-h(Y,σ) by $\Psi_f(h)=fohof^{-1}$ for every h $\in$ #rgc-h(X,τ). Then Ψ_f is a bijection. Further, for all h_1,h_2 $\in$ #rgc-h(X,τ);

$$\Psi_f(h_1oh_2) = fo(h_1oh_2)of^{-1}$$
$$= (foh_1of^{-1})o(foh_2of^{-1})$$
$$= \Psi_f(h_1) \text{ o } \Psi_f(h_2) .$$

Hence Ψ_f is a homomorphism and so it is an isomorphism induced by f.

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