Secondary k-Generalized Inverse of a s-k-Normal Matrices

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Abstract:

Secondary k-generalized inverse of a given square matrix is defined and its characterizations are given. Secondary k- generalized inverses of s-k normal matrices are discussed.

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1.Introduction:

Ann Lee initiated the study of secondary symmetric matrices in[1]. The concept of secondary k - normal matrices was introduced in [3]. Some equivalent conditions on secondary k- normal matrices are given in [4]. In this paper we describe secondary k- generalized inverse of a square matrix, as the unique solution of a certain set of equation . This secondary k-generalized inverse exists for particular kind of square matrices. Let C_{nxn} denote the space of nxn complex matrices. We deal with secondary k-generalized inverse of s-k normal matrices. Throught this paper, if $A \in C_{nxn}$, then we assume that if $A \neq 0$ then $A(KVA^{*}VK) \neq 0$

i.e., $A(KVA^{*}VK) = 0 \quad \Rightarrow A = 0 \quad \rightarrow \quad (1)$

It is clear that the conjugate secondary k transpose satisfies the following properties.

$$KV(A+B)^{*}VK = (KVA^{*}VK) + (KVB^{*}VK)$$

$$KV(\lambda A)^{*}VK = \bar{\lambda}(KVA^{*}VK)$$

$$KV(BA)^{*}VK = (KVA^{*}VK)(KVB^{*}VK)$$

Now if $BA(KVA^{*}VK) = CA(KVA^{*}VK)$ then by (1)

$$BA(KVA^{*}VK) - CA(KVA^{*}VK) = 0$$

$$\Rightarrow (BA(KVA^{*}VK) - CA(KVA^{*}VK))(KV(B-C)^{*}VK) = 0$$

$$\Rightarrow (BA-CA)(KV(BA-CA)^{*}VK) = 0$$

$$\Rightarrow (BA-CA) = 0$$

$$\Rightarrow BA = CA$$

Therefore $BA(KVA^{*}VK) = CA(KVA^{*}VK) \Rightarrow BA = CA \rightarrow$ (2)

Similarly, $B(KVA^{*}VK)A = C(KVA^{*}VK)A$

$$\Rightarrow B(KVA^{*}VK) = C(KVA^{*}VK) \rightarrow (3)$$

Definition 1.1: [3]

A Matrix $A \in C_{nxn}$ is said to be secondary k-normal (s-k normal) if

$$A(KVA^{*}VK) = (KVA^{*}VK)A$$

Example 1.2:

$A = \begin{pmatrix} i & 2 & 3 & 4 \\ 4 & -i & 2 & 3 \\ 3 & 4 & i & 2 \\ 2 & 3 & 4 & -i \end{pmatrix}$ is a s-k normal matrix for k=(1,3),(2,4) the permutation matrix be

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Definition 1.3:

A matrix $A \in C_{nxn}$ is said to be secondary k-unitary (s-k unitary) if

$$A(KVA^{*}VK) = (KVA^{*}VK)A = I$$

Example 1.4:

$A= \begin{pmatrix} i & -1 & 1 & 0 \\ -1 & -i & 0 & 1 \\ 1 & 0 & i & -1 \\ 0 & 1 & -1 & -i \end{pmatrix}$ is a s-k unitary matrix

Section 2: Secondary k - Generalized inverses of a matrix

Theorem 2.1:

For any $A \in C_{nxn}$, the four equations

$$AXA = A \quad \rightarrow (4)$$

$$XAX = X \quad \rightarrow (5)$$

$$KV(AX)^{*}VK = AX \quad \rightarrow (6)$$

$$KV(XA)^{*}VK = XA \quad \rightarrow (7)$$

have a unique solution for any $A \in C_{nxn}$.

Proof: First, we shall show that equations (5) & (6) are equivalent to the single equation

$$XKV(AX)^{*}VK = X \quad \rightarrow (8)$$

From equations (5) and (6), (7) follows, since it is merely (6) substituted in (5) Conversely, equation (8) implies $AXKV(AX)^{*}VK = AX$

Since the left hand side is s-k hermitian, (6) follows. By substituting (6) in (8), we get $XAX = X$ which is actually (5). Therefore (5) and (7) are equivalent to (8) Similarly, (4) & (7) are equivalent to the equation

$$XA(KVA^{*}VK) = KVA^{*}VK \rightarrow (9)$$

Thus to find a solution for the given set of equations, it is enough to find an X satisfying (8) & (9). Now the expressions $((KVA^{*}VK)A), ((KVA^{*}VK)A)^{2}, ((KVA^{*}VK)A)^{3} \ldots$ cannot all be linearly independent (i.e) there exists a relation

$$\lambda_{1}((KVA^{*}VK)A) + \lambda_{2}((KVA^{*}VK)A)^{2} + \ldots + \lambda_{k}((KVA^{*}VK)A)^{k} = 0 \rightarrow (10)$$

Where $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ are not all zero. Let λ_{r} be the first non zero λ. (i.e) $\lambda_{1} = \lambda_{2} = \ldots \lambda_{r-1} = 0$. Therefore (10) implies that

$$\lambda_{r}((KVA^{*}VK)A)^{r} = -\left\{\lambda_{r+1}((KVA^{*}VK)A)^{r+1} + \ldots + \lambda_{m}((KVA^{*}VK)A)^{m}\right\}$$

If we take $B = -\lambda_{r}^{-1}\left\{\lambda_{r+1}I + \lambda_{r+2}((KVA^{*}VK)A) + \ldots + \lambda_{m}((KVA^{*}VK)A)^{m-r-1}\right\}$

Then

$$B((KVA^{*}VK)A)^{r+1} = -\lambda_{r}^{-1}\left\{\lambda_{r+1}((KVA^{*}VK)A)^{r+1} + \ldots + \lambda_{m}((KVA^{*}VK)A)^{m}\right\}$$

$B((KVA^{*}VK)A)^{r+1} = ((KVA^{*}VK)A)^{r}$. By using (2) & (3) repeatedly, we get

$$B(KVA^{*}VK)A(KVA^{*}VK) = KVA^{*}VK \quad \rightarrow (11)$$

Now if we take $X = B(KVA^{*}VK)$ then (11) implies that this X satisfies (9)

implies (7), we have $(KV(XA)^{*}VK)(KVA^{*}VK) = KVA^{*}VK$

$$\Rightarrow B(KV(XA)^{*}VK)(KVA^{*}VK) = B(KVA^{*}VK)$$

Therefore $X = B(KVA^{*}VK)$ satisfies (8). Thus $X = B(KVA^{*}VK)$ is a solution for the given set of equations.

Now let us prove that this X is unique. Suppose that X and Y satisfy (8) and (9). Then by substituting (7) in (5) and (6) in (4), we obtain

$(KV(XA)^{*}VK)X = X$ and $(KV(AX)^{*}VK)A = A$

Also, $Y = (KV(YA)^{*}VK)Y$ and $KVA^{*}VK = (KVA^{*}VK)AY$

Now $X = X(KVX^{*}VK)(KVA^{*}VK)$

$$= X(KVX^{*}VK)(KVA^{*}VK)AY$$

$$= X(KV(AX)^{*}VK)AY$$

$$= XAY$$

$$= XA(KV(YA)^{*}VK)Y$$

$$= XA(KVA^{*}VK)(KVY^{*}VK)Y$$

$$= (KVA^{*}VK)(KVY^{*}VK)Y$$

$$= (KV(YA)^{*}VK)Y$$

$$X = Y$$

Therefore X is unique.

Definition 2.2: Let $A \in C_{nxn}$. The unique solution of (4), (5), (6) and (7) is called secondary k-generalized inverse of A and is written as $A^{\dagger sk}$.

Example 2.3:

If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ then $A^{\dagger sk} = \begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}$

Note 2.4: By using (7) in (5), (6) in (4) and from (8) and (9) we obtain

$$\left.\begin{aligned} & A^{\dagger sk}(KV(A^{\dagger sk})^{*}VK)(KVA^{*}VK) = A^{\dagger sk} = (KVA^{*}VK)(KV(A^{\dagger sk})^{*}VK)A^{\dagger sk} \\ & A^{\dagger sk}A(KVA^{*}VK) = (KVA^{*}VK) = (KVA^{*}VK)AA^{\dagger sk} \end{aligned}\right\} \rightarrow \qquad (12)$$

If λ is a scalar, then $\lambda^{\dagger sk}$ means λ^{-1} when $\lambda \neq 0$ and zero when $\lambda = 0$.

Section3: Secondary- k-generalized inverse of s-k normal matrices.

In this paper, characterizations of secondary k-generalized inverse (s-k-g) inverse of a matrix are obtained s-k-g inverse of s-k-normal matrices are discussed .s-k herimitian matrices are defined and the condition for s-k normal matrices to be diagonal is investigated.

Theorem 3.1: For $A \in C_{nxn}$.

(i) $\left(A^{\dagger sk}\right)^{\dagger sk} = A$ (ii) $\left(KV\left(A^{*}\right)^{\dagger sk}VK\right) = \left(KV\left(A^{\dagger sk}\right)^{*}VK\right)$

(iii) If A is non singular, then $A^{\dagger sk} = A^{-1}$ (iv) $(\lambda A)^{\dagger sk} = \lambda^{\dagger sk}A^{\dagger sk}$

(v) $((KVA^{*}VK)A)^{\dagger sk} = A^{\dagger sk}(KVA^{\dagger sk}VK)^{*}$

Proof: Let $A \in C_{nxn}$.

(i) By the definition of s-k-g inverse, we have

$$A^{\dagger sk} A A^{\dagger sk} = A^{\dagger sk} \text{ and } A^{\dagger sk} (A^{\dagger sk})^{\dagger sk} A^{\dagger sk} = A^{\dagger sk}$$

These two equations imply that

$$A^{\dagger sk \dagger sk} = A$$

(ii) From the definition of $A^{\dagger sk}$, we have $AA^{\dagger sk} A = A$

$$\Rightarrow (KVA^{*}VK)(KV(A^{\dagger sk})^{*}VK)(KVA^{*}VK) = KVA^{*}VK$$

Also $(KVA^{*}VK)(KV(A^{*})^{\dagger sk}VK)(KVA^{*}VK) = KVA^{*}VK$

From these two equations, we have

$$(KV(A^{\dagger sk})^{*}VK) = KV(A^{*})^{\dagger sk}VK$$

(iii) Since A is non singular, A^{-1} exists

Now $AA^{\dagger sk} A = A$ (By definition of $\dagger_{sk}$)

Pre multiplying & post multiplying by A^{-1} we have

$$A^{\dagger sk} = A^{-1}$$

(iv) The equations, $AA^{\dagger sk} A = A$ and

$(\lambda A)(\lambda A)^{\dagger sk}(\lambda A) = (\lambda A)$ imply that

$$\lambda(\lambda A)^{\dagger sk} = A^{\dagger sk}$$

$$\Rightarrow (\lambda A)^{\dagger sk} = \lambda^{\dagger sk} A^{\dagger sk} \text{ where } \lambda^{\dagger sk} = \lambda^{-1}$$

(v) from (12) we have,

$$A^{\dagger sk}(KV(A^{\dagger sk})^{*}VK)(KVA^{*}VK) = A^{\dagger sk}$$

Also $AA^{\dagger sk}A = A$.

Therefore $AA^{\dagger sk}(KV(A^{\dagger sk})^{*}VK)(KVA^{*}VK)A = A$

Substitute this in the right hand side of the defining relation, we get

$$((KVA^{*}VK)A)^{\dagger sk} = A^{\dagger sk}(KV(A^{*})^{\dagger sk}VK)$$

Theorem 3.2: A necessary and sufficient condition for the equation $AXB = D$ to have a solution is $AA^{\dagger sk}DB^{\dagger sk}B = D$, in which case the general solution is

$X = A^{\dagger sk}DB^{\dagger sk} + Y - A^{\dagger sk}AYBB^{\dagger sk}$, where Y is arbitrary.

Proof: Let us assume that X satisfies the equation $AXB = D$, then $D = AXB$ $= AA^{\dagger sk}AXBB^{\dagger sk}B = AA^{\dagger sk}DB^{\dagger sk}B$ (By the definition of $\dagger_{sk}$)

Conversely if $D = AA^{\dagger sk}DB^{\dagger sk}B$, then $X = A^{\dagger sk}DB^{\dagger sk}$, then it is a particular solution of $AXB = D$. since $AXB = AA^{\dagger sk}DB^{\dagger sk}B = D$.

If $Y \in C_{nxn}$, then any expression of the form $X = A^{\dagger sk}DB^{\dagger sk} + Y - A^{\dagger sk}AYBB^{\dagger sk}$ is a solution of $AXB = D$.and conversely,if X is a solution $AXB = D$, then $X = A^{\dagger sk}DB^{\dagger sk} + X - A^{\dagger sk}AXB^{\dagger sk}B$ satisfies $AXB = D$. Hence the theorem.

Theorem 3.3: The matrix equations $AX = B$ and $XD = E$ have a common solution if and only if each equation has a solution and $AE = BD$.

Proof: It is easy to see that the conditions is necessary, conversely $A^{\dagger sk}B$ and $ED^{\dagger sk}$ are solutions of AX=B and XD=E and hence $AA^{\dagger sk}B = B$ and $ED^{\dagger sk}D = E$. Also AE=BD. By using these facts it can be prove that $X = A^{\dagger sk}B + ED^{\dagger sk} - A^{\dagger sk}AED^{\dagger sk}$ is a common solution of the given equations.

Definition 3.4: A matrix $E \in C_{nxn}$ is said to be secondary-k hermitian idempotent matrix (s-k. h.i) if $E(KVE^{*}VK) = E$ (i.e) $E = KVE^{*}VK$ and $E^{2} = E$.

Theorem 3.5: (i) $A^{\dagger sk}A$, $AA^{\dagger sk}$, $1\text{-}A^{\dagger sk}A$, $1\text{-}AA^{\dagger sk}$ are all the s-k hermitian idempotent.

(ii) J is idempotent ⇔ there exist s-k hermitian idempotent's E and F such that $J = (FE)^{\dagger sk}$ in which case J = EJF.

Proof: Proof of (i) is obvious. If J is idempotent then J^2=J. By (i) of theorem (3.1), $J = \left\{(J^{\dagger sk}J)(JJ^{\dagger sk})\right\}^{\dagger sk}$. Now if we take $E = JJ^{\dagger sk}$ and $F = J^{\dagger sk}J$ they will satisfy our requirements conversely if $J = (FE)^{\dagger sk}$ then J=EFPEF where

$P = (KV((FE)^{\dagger sk})^{*}VK)(FE)^{\dagger sk}(KV(FE)^{\dagger sk})^{*}VK)$. Therefore J=EJF and hence

$J^{2} = E(FE)^{\dagger sk}\ FE(FE)^{\dagger sk}\ F = E(FE)^{\dagger sk}F = J$. Hence J is idempotent.

Note 3.6:(i) s-k hermitian idempotent matrices are s-k normal matrices.

(ii) The s-k-g inverse of an s-k hermitian idempotent matrix is also s-k hermitian idempotent matrix.

Definition 3.7: For any square matrix A there exists a unique set of matrices J_{λ} defined for each complex number λ such that

$$J_{\lambda}J_{\mu} = \delta_{\lambda\mu}J_{\lambda} \quad \rightarrow (13)$$

$$\sum J_{\lambda} = 1 \quad \rightarrow (14)$$

$$AJ_\lambda = J_\lambda A \quad \rightarrow (15)$$

$$(A-\lambda I)J_\lambda \text{ is nilpotent} \rightarrow (16)$$

Then the non zero J_λ 's are called the principal idempotent elements of A.

Theorem 3.8: If $E_\lambda = 1-\{(A-\lambda I)^n\}^{\dagger sk}(A-\lambda I)^n$ and $F_\lambda = 1-(A-\lambda I)^n\{(A-\lambda I)^n\}^{\dagger sk}$,

where n is sufficiently large, then the principal idempotent element of A are given by $J_\lambda = \{F_\lambda E_\lambda\}^{\dagger sk}$ and n can be taken as unity iff A is diagonable.

Proof: Assume that A is diagonable.

Let $E_\lambda = 1-\{(A-\lambda I)\}^{\dagger sk}(A-\lambda I)$, and $F_\lambda = 1-(A-\lambda I)\{(A-\lambda I)\}^{\dagger sk}$

Then by 3.5(i) E_λ and F_λ are s-k hermitian idempotent matrices. If λ is not an eigen value of A, then $|A-\lambda I| \neq 0$ and hence F_λ and E_λ are zero by (iii) of theorem(3.1). Clearly, $(A-\mu I)E_\mu = 0$ and $F_\lambda(A-\lambda I) = 0 \rightarrow (17)$

Therefore $\mu F_\lambda E_\mu = F_\lambda A E_\mu = \lambda F_\lambda E_\mu$

Hence $F_\lambda E_\mu = 0$ if $\lambda \neq \mu \rightarrow (18)$

Now if we take $J_\lambda = \{F_\lambda E_\lambda\}^{\dagger sk}$ then by (ii) of theorem (3.5),

$$J_\lambda = E_\lambda\{F_\lambda E_\lambda\}^{\dagger sk}F_\lambda \quad \rightarrow (19),$$

Now(18) implies $J_\lambda J_\mu = \delta_{\lambda\mu}J_\lambda$ .Also by(18), $F_\lambda J_\mu E_\gamma = \delta_{\lambda\mu}\delta_{\mu\gamma}F_\lambda E_\lambda \rightarrow (20)$ If Z_α is an eigen vector of A corresponding to the eigen value α then $E_\alpha Z_\alpha = Z_\alpha$. As A is diagonable, any column vector X conformable with A is expressible as a sum of eigen vectors (i.e) it is expressible in the form $X = \sum E_\lambda X_\lambda$. This is a finite sum over all complex λ.

Similarly, if Y^* is conformable with A, it is expressible as $Y^* = \sum Y_\lambda^* F_\lambda$ Now by equations (18) and (20) $Y^*(\sum J_\mu)X = (\sum Y_\lambda^* F_\lambda)(\sum J_\mu)(\sum E_\gamma X_\lambda)$

$$= \sum Y_\lambda^* F_\lambda E_\lambda X_\lambda$$

$$= Y^*(\sum J_\mu)X = Y^* X$$

$$\Rightarrow Y^*(\sum J_\mu - I)X = 0$$

$$\Rightarrow \sum J_\mu = I$$

Also(17) and (19) lead to $AJ_\lambda = \lambda J_\lambda = J_\lambda A \rightarrow (21)$

This implies $(A - \lambda I)J_\lambda$ is nilpotent and (15) and (16) are satisfied.

Moreover $A = \sum \lambda J_\lambda \qquad \rightarrow (22)$

Conversely if $\sum J_\lambda = I$ and A is not diagonable (n=1) then $X = \sum J_\lambda X$ gives X as a sum of eigen vectors of A, since (21) was derived without assuming the diagonability of A. If A is not diagonable. It seems more convenient simply to prove that for any set of J_λ's satisfying (19), (20), (21) & (22) each $J_\lambda = (F_\lambda E_\lambda)^{\dagger sk}$ where F_λ and E_λ are defined as in the theorem.

If the J_λ's satisfy (13), (14), (15) & (16) $\sum J_\lambda = I$ and

$$(A - \lambda I)^n J_\lambda = 0 = J_\lambda (A - \lambda I)^n \qquad \rightarrow (23)$$

Which comes by using the fact that $(A - \lambda I)J_\lambda$ is nilpotent, where n is sufficiently large. From (23) and the definition of E_λ and F_λ, we have

$$E_\lambda F_\lambda = J_\lambda = J_\lambda F_\lambda \qquad \rightarrow (24)$$

By using Euclid's algorithm, there exist P and Q which are polynomials in A such that $I = (A-\lambda I)^n P + Q(A-\mu I)^n$ if $\lambda \neq \mu$.

Now $F_\lambda (A-\lambda I)^n = 0 = (A-\mu I)^n E_\mu$. Hence $F_\lambda E_\mu = 0$ if $\lambda \neq \mu$

From (24) $F_\lambda J_\mu = 0 = J_\lambda E_\mu$ if $\lambda \neq \mu$. Since $\sum J_\mu = I$, we get

$F_\lambda J_\lambda = F_\lambda$ *and* $J_\lambda E_\lambda = E_\lambda \rightarrow$(25) using (24) and (25) it is easy to see that

$\{F_\lambda E_\lambda\}^{\dagger s\text{-}k} = J_\lambda$. Hence the theorem

Theorem 3.9: If A is s-k normal, it is diagonable and its principal idempotent elements are s-k hermitian.

Proof: If A is s-k normal then $(A-\lambda I)$ is s-k normal. By using (viii)of the theorem (3.1) in the definition of E_λ and F_λ of theorem (3.8) we obtain $E_\lambda = 1-\{(A-\lambda I)\}^{\dagger sk}(A-\lambda I)$ and

$F_\lambda = 1-(A-\lambda I)\{(A-\lambda I)\}^{\dagger sk}$

Hence A is diagonable. since $E_\lambda = F_\lambda$, $J_\lambda = E_\lambda$ is s-k hermitian.

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